On thermodynamic active control of shape memory alloy wires

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Received 20 September 2001; received in revised form 6 June 2002; accepted 19 June 2002

This paper is dedicated to the memory of the late J.L. Lions, whose seminal work in applied mathematics led, in particular, to the embedding of control-theoretic problems in the context of the modern theory of partial differential equations and inspired us all.

Abstract

We propose a model for the control of high-frequency oscillations in shape memory alloy wires. We introduce a notion of generalized solution for a generalized control and in this context prove a local exact controllability result effectively corresponding to an approximate controllability result for the nonconvex pseudoelastic material system.

Keywords: Microstructures; Dynamic control; Generalized control; Degenerate wave equation

1. Introduction

We consider the time-dependent modeling of shape memory alloy (SMA) wires from a control-theoretic viewpoint. Our principal control-theoretic result is one of local exact controllability in a context of generalized solutions for a model of temperature-controlled SMA wires, which we make precise in Definition 3.3. While investigation is still in progress, this result suggests approximate controllability for the exact model with feasibly implemented control action.

The issue of active controllability of SMAs has received little attention so far, even though it constitutes an emerging microscale technology, primarily because the underlying thermodynamics has been developed only recently. We refer to [3] for a more complete overview of the thermodynamics of phase boundaries in complex materials, see also [16]. We focus in this communication on controlling the dynamics of SMAs using temperature variations. We are concerned with issues of controllability in a context that has many important applications, including, e.g., the control of microvalves and micropumps medical space applications [19,20].

Any ability to control and to provide stabilization using SMA systems resides in the possibility to
initiate and control the motion of phase boundaries in these composites. SMAs can exist in two different temperature-dependent phases: a lower temperature phase, Martensite, and a higher temperature phase, Austenite. Our goal in this paper is to model a setting involving the rapid alternation of these phases and to consider local controllability in the neighborhood of an equilibrium of that system. We wish to find a plausible mechanism for distributing temperature variations such that, for given initial and terminal states of a SMA wire, each close to a special equilibrium, there will exist a control $\chi = \gamma(x,t)$ of the temperature such that we reach the specified terminal state at the specified time $T$. While ultimately one must deal with heat transfer in considering the (approximate) implementation of this temperature control, for present purposes we simply take the pointwise temperature as perfectly specifiable as a function of $x,t$. Somewhat artificially we only distinguish between ‘higher’ and ‘lower’ temperatures as such, restricting our attention to the use of only two temperatures. This restriction implies an underestimation of the possibility of control, but our acceptance of generalized solutions corresponding to highly oscillatory limits (effectively couched in a context of Young measures) somewhat compensates for this.

2. A sketch of the thermodynamic modeling

2.1. Phases

The existence of two phases in SMAs may be understood through consideration of the Helmholtz free energy $\Phi$ for these materials: as with most natural systems, the crystal lattice of SMA changes in order to minimize this energy. In Fig. 1 we see a sketch of the one-dimensional free energy $\Phi$ as a function of the lattice deformation gradient $\gamma$ (i.e., $\gamma$ denotes the strain) and the temperature $\theta$. Here, the reference configuration is that of the Austenite phase, with strain $\gamma = 0$; this is seen to be the stable phase for temperatures $\theta \geq \theta_A$. By tracking the changes in the plots as the temperature decreases from $\theta_A \geq \theta_M$, we then see how the Martensitic phase transformation occurs. Once the temperature has reached $\theta \leq \theta_M$, we see that the metastable Austenite state corresponding to $\gamma = 0$ has disappeared, and has been replaced by the two equivalent stable Martensite states corresponding to $\gamma = \pm \gamma_0$.

This transformation to a Martensite phase is locally determined, depending on the temperature at the material point $x$. Since various parts of the SMA may not have the same temperature, the Austenite and Martensite phases may simultaneously both exist at different (possibly nearby) locations, and, of course, may vary in time. Of special importance in the use of SMAs for control of vibrational damping are the boundaries between these phases, where the crystal transformation occurs. The movement of this phase boundary can act as the active mechanism for a stabilizing conversion of mechanical energy to heat energy, cf., [1,2,4].

The Austenitic free energy function $\gamma \mapsto W_0$ (e.g., corresponding to some typical temperature $\theta_0 > \theta_A$) is a single-well potential and the Martensitic free energy function $W_1(\cdot)$ at $\theta_1 < \theta_M$ is a double-well potential. Although the exact choices of $W_0, W_1$ are not really
significant for our present purposes, we note that fairly standard modeling versions of these are

\[ \mathcal{W}_0(\gamma) = \frac{1}{2} \gamma^2 \]  
for the Austenite phase, \hspace{1cm} (1a)

\[ \mathcal{W}_1(\gamma) = \frac{1}{4} (\gamma^2 - \gamma_0^2)^2 \]  
for the Martensite phase. \hspace{1cm} (1b)

### 2.2. Thermodynamics

We begin by modeling the thermodynamics of SMAs, coupling mechanics and heat transfer. Fischer et al.\cite{11} and Bekker et al.\cite{10} model the mechanics of SMA using a linear wave equation having linear stress–strain relationship \( \sigma = \sigma(\gamma) \). Abeyaratne and Knowles\cite{1,2,4}, Luskin\cite{17} and Ball et al.\cite{9}, however, do not allow linear simplification of the stress–strain relationship. Klouček and Luskin\cite{15} and Klouček\cite{14} have further incorporated viscous stresses into the dynamical equation so the viscoelastic equation reads

\[
\frac{\partial^2 u(x,t)}{\partial t^2} = \text{div} \left( \epsilon \mathcal{W}'(\gamma(x,t), 0(x,t)) \right) + A(x,t) \frac{\partial \gamma(x,t)}{\partial t}, \quad x \in \Omega, \ t \geq t_0. \hspace{1cm} (2)
\]

The strain \( \gamma \) denotes just \( \nabla u : \mathbb{R}^n \to \mathbb{R}^n, \ n = 1, 2, 3 \). Here, \( A \in \mathbb{R}^{n \times n} \) is the viscosity coefficient matrix and \( \mathcal{W}' : \mathbb{R}^{n \times n} \times \mathbb{R}^n \to \mathbb{R}_+ \) is the referential strain energy density, e.g., as given by the quadruple-well Ericksen–James energy density describing cubic-to-tetragonal phase transformations (from\cite{14,15,17}) so \( \gamma \mapsto \epsilon \mathcal{W}' \) is the constitutive stress/strain relation. This is coupled with a nonlinear heat conduction equation, from Klouček\cite{15},

\[
\frac{\partial \theta}{\partial t} = \theta \text{Tr} \left( \frac{\partial^2 \mathcal{W}}{\partial \gamma \partial \theta} (\gamma, \theta) \frac{\partial \gamma}{\partial t} \right) + \text{Tr} \left( \frac{\partial \gamma}{\partial t} A \frac{\partial \gamma}{\partial t} \right) + \text{div} \left( \gamma C^{-1} \nabla \theta + r(x,t) \right),
\]

\[ x \in \Omega, \ t \geq t_0. \hspace{1cm} (3)
\]

Eqs. (2) and (3) govern the thermodynamics of SMAs. We note that the term \( \text{div} \text{det}(\gamma C^{-1} \nabla \theta) \) represents a generalization of the standard Fourier law for the heat flux by incorporating an interplay between the right Cauchy–Green tensor and the temperature gradient.

### 2.3. Some solvability results

The global existence of regular solutions for (2) is a delicate and complicated issue. Typical assumptions guaranteeing the existence (and uniqueness) of solutions for nonlinear wave equations usually involve convexity of \( \mathcal{W}' \), although, e.g., the case of nonconvex \( \mathcal{W}' \) (including nonlinear viscosity with higher order growth, \( \mathcal{W}'(\nabla u) \sim |\nabla u|^p, p > 2 \)) is treated in\cite{12}. If the convexity assumption is to be dropped, then one must assume Lipschitz continuity for the stress \( D\mathcal{W}'(\cdot) \)\cite{18}. A global existence result is obtained in\cite{13} under the assumption of monotonicity of the stress at infinity, proposed in\cite{5}, but this result relies on the fact that the viscosity is there a linear operator, although one might note that such linearity may be inconsistent with the frame indifference of the stress and so violate basic physics; this is discussed in detail in\cite{6,7,8}. The following result from\cite{13} applies to (2) quite generally.

**Theorem 2.1.** Let \( \Omega \subset \mathbb{R}^n \) be bounded and open. Let us assume that the energy density \( \mathcal{W}' \) satisfies the assumptions

\[
\mathcal{W}' \in C^2(M^{N \times N}) ,
\]

\[
C_1 |s|^p - 1 \leq \mathcal{W}'(s) \leq C_2 (|s|^p + 1),
\]

\[
C_1, C_2 > 0, \ p \geq 2,
\]

\[
(D\mathcal{W}'(s_1) - D\mathcal{W}'(s_2)) \cdot (s_1 - s_2) \geq - K |s_1 - s_2|, \hspace{1cm} (4)
\]

and that the matrix \( A = A(x,t) \) in (2) is independent of the space and time variables. Then there exists a unique weak solution \( u \) of (2) such that

\[
u \in L^\infty((0, \infty); \{ v \in W^{1,p}(\Omega) \mid v - g \in W^{1,p}_0(\Omega) \}),
\]

\[
\cap W^{1,\infty}((0, \infty); L^2(\Omega))
\]

\[
\cap W^{1,2}_{\text{Loc}}([0, \infty); W^{1,2}(\Omega))
\]

\[
\cap W_{L^\infty}^{2,2}([0, \infty); W^{-1,q}(\Omega)), \quad 1/p + 1/q = 1. \hspace{1cm} (5)
\]
provided the Dirichlet boundary condition \( g \) is in \( W_0^{1,p}(\Omega) \), the initial condition \( u_0 \) for the velocity \( u \), is in \( L^2(\Omega) \), and the initial deformation \( u_0 \) is in \( W^{1,p}(\Omega) \). Then

\[
\int_0^\infty \int_\Omega \gamma(t,x) \nabla \varphi(x,t) \\
+ D \mathcal{W}(\nabla u(x,t)) \nabla \varphi(x,t) \, dx \, dt \\
= \int_0^\infty \int_\Omega u_t(x,t) \varphi(x,t) \, dx \, dt
\]

for all \( \varphi \in C^\infty(\Omega \times (0,\infty)) \)

(6)

and \( u(x,t)|_{t=0} = u_0(x) \), \( u_t(x,t)|_{t=0} = u_1(x) \) in \( \Omega \).

There does not exist to our knowledge a similar statement for the complete thermodynamical model (2)–(3), augmented with suitable initial and boundary conditions. We note that the numerical calculations in Klouček [14] exhibit a rather complicated intertwining of the heat flux and the lattice deformation and that the numerical results presented there suggest that the system may not always have a smooth solution.

3. Control modeling

3.1. A simplified model

In order to avoid the difficulties described above, and in order to initiate an examination of the controllability of SMA systems, we radically simplify models (2) and (3).

The first simplification consists of reduction to a one-dimensional model instead of the full three-dimensional version, confining ourselves to the setting of SMA wire systems, with length scale much greater than diameter. This will mean (say, for the Austenite phase) that we effectively take the strain to be simply \( \gamma = \hat{\gamma} u \) where \( u \) is (pointwise) the deviation in position from the reference position \( x \), where we are assuming that the wire is in equilibrium when one uses this reference position as coordinization.

More significantly, a second main simplification of our model is that, somewhat unphysically, we will consider only the mechanics as in (2), with just two possible temperatures \( \theta_0 \) and \( \theta_1 \), respectively, corresponding to the Austenite and Martensite phases, and will then take the pointwise choice between these temperatures as a control, ignoring the dynamics of heat transfer through the wire as in (3). Thus, the control \( \chi = \chi(x,t) \in \{0,1\} \) identifies a choice of temperature \( \theta(x,t) \in \{\theta_0,\theta_1\} \). This simplification will provide us with our first step in the control process. In doing this we will indeed lose the thermodynamics of the system and consideration of the relevant time scale for the temperature variation. We have implicitly assumed here that the time scale and control mechanism permits rapid temperature fluctuations so we may (fairly arbitrarily) “choose” \( \chi = \chi(x,t) \) pointwise. Note that the relaxation time for adaptation of the crystallographic structure to a new temperature is already implied by (2). All of this is perhaps plausible in spirit, but is certainly false in detail, so we may get an overly optimistic picture of what is possible. Nevertheless, we do anticipate that the new model should still provide a rough approximation to certain possible controlled system behaviors.

Under these simplifications we change notation slightly: our strain energy density is now \( \mathcal{W} = \mathcal{W}(\cdot,\chi) = \mathcal{W}(\chi(x,t),\cdot) \), given for \( \chi = 0 \) or 1 by

\[
\mathcal{W}(\gamma,\chi) = \chi \mathcal{W}(\gamma) + (1-\chi) \mathcal{W}(0)
\]

(7)

where \( \mathcal{W}(0) \) is the Austenitic free energy function which used to be denoted by \( \mathcal{W}(\gamma,\theta_0) \) and, similarly, \( \mathcal{W}(1) \) is the Martensitic free energy function formerly denoted by \( \mathcal{W}(\gamma,\theta_1) \); compare (1). Note that we have not yet defined \( \mathcal{W}(\gamma,\chi) \) for intermediate values \( 0 < \chi < 1 \) and certainly this will not be what would previously have been denoted by \( \mathcal{W}(\gamma,\chi\theta_1 + (1-\chi)\theta_0) \).

Using (7), we can now write the viscoelastic system (2) in the form

\[
u_t(x,t) = \hat{\gamma}(\gamma(x,t) + \nu v_t(x,t))
\]

with \( \gamma(x,t) \) given as in (7) and with \( \nu > 0 \).

At present, we cannot use Theorem 2.1 to treat existence of solutions of (8) for the step function \( \chi \), but we do note that, if the problem were regularized
and \( \chi \) would be a twice differentiable function, then there would necessarily exist a global weak solution of (8). This result, stated in the following Lemma, is an immediate consequence of Theorem 2.1.

**Lemma 3.1.** Assume \( \chi \in C^2((0,1) \times (0,T)) \), and let \( u_0, u_1 \in W^{1,4}(0,1) \). We take \( w_0, w_1 \) as in (1). Then there exists a unique global weak solution of (8) having the properties (5).

**Proof.** In order to use Theorem 2.1 we must verify conditions (4) for the one-dimensional strain energy density \( W \) from Eq. (7), along with the proper initial and boundary conditions.

The initial and boundary conditions from Theorem 2.1 hold by assumption, using the choice \( p = 4 \). Further, in our consideration of (8) we may safely assume, for use within the proof of the lemma, the bounds

\[
\gamma(x,t) \in (-1,1), \quad \gamma_0 \in (0,1), \quad \chi(x,t) \in [0,1].
\]  

(9)

Now, \( W \in C^2 \) since \( \chi \in C^2((0,1) \times (0,T)) \), by assumption, and

\[
\frac{\partial^2}{\partial \gamma^2} W(\gamma, \chi) = \chi(3\gamma^2 - \gamma_0^2 - 1) + 1 \in C^0.
\]  

(10)

Using bounds (9), we have (with \( p = 4 \) as mentioned above) that

\[
|\gamma|^p - 1 \leq W(\gamma, \chi) \leq |\gamma|^p + 1
\]  

(11)

for all allowable \( \gamma, \gamma_0 \) and \( \chi \).

Finally, the third requirement from (4) for problem (8) becomes, for some \( K > 0 \), that

\[
(DW(\gamma_1) - DW(\gamma_2)) \cdot (\gamma_1 - \gamma_2) \geq -K|\gamma_1 - \gamma_2|^2
\]

\[
(4\chi(\gamma_1^2 + \gamma_2^2) - 4\chi(x,t)\gamma_0^2 + 1 - \chi)(\gamma_1^2 - \gamma_2^2)
\]

\[
\geq -K(\gamma_1 - \gamma_2)^2.
\]  

(12)

Since, in view of (9),

\[
4\chi(\gamma_1^2 + \gamma_2^2) - 4\gamma_0^2 + 1 - \chi
\]

\[
\geq -4\gamma_0^2 + 1 - \chi \geq -4,
\]

inequality (12) is satisfied with \( K = 4 \).

Thus, the energy density \( W \) for problem (8) satisfies the requirements of the Theorem 2.1, and the result follows. \( \square \)

3.2. Generalized solutions

Our next step is to consider a suitably generalized notion of solutions which corresponds in some sense to a limit of the “two-temperature” setting. Let us note that when we switch values for the temperature at some material point, the energy density jumps by \( \pm[\partial_\gamma W_1 - \partial_\gamma W_0]u_x \) at that point. Thus, our assumption that we can change the temperature arbitrarily can be expected to defeat any suggestion of an intrinsic mechanism for bounding the energy. However, we do note that rapidly oscillating temperature fluctuations (rapid alternation of these temperature switchings) would tend to cancel in their resultant effect on energy; compare the classical Riemann–Lebesgue Lemma of real analysis: \( \int_a^b f(x) \sin nx \, dx \to 0 \) as \( n \to \infty \) for any integrable function \( f \). Thus, the choice of a rapidly oscillating control \( \gamma \) may plausibly produce a finite energy solution, even a smooth solution, in an appropriate limit.

What we want is to permit replacing \( \chi \) as our control function by a smoother function \( \tilde{\chi} \), allowed to take more general values in \([0,1] \), corresponding to \( \chi \to \tilde{\chi} \) for some approximating sequence of \([0,1] \)-valued controls \( \chi \). Let us describe the nature of such a limit, which we will then consider to be a generalized solution of model (8). The meaning of the generalized solution will be made precise in Definition 3.3.

As is usual, we may reasonably consider, near any material point \( x \), a modeling domain having a characteristic length scale \( h \); we then let \( \tilde{\chi}_h = \tilde{\chi}_h(x,t) \) be the fraction of this domain, with respect to length in the reference coordinates, for which the temperature is \( \partial_1 \), so \( 1 - \tilde{\chi}_h \) is the complementary fraction for which the temperature is \( \partial_0 \). Thus, \( \tilde{\chi}_h \) is the local average of \( \chi \), i.e.,

\[
\tilde{\chi}_h(x,t) = \frac{1}{h} \int_x^{x+h} \chi(\xi,t) \, d\xi.
\]  

(13)

In this sense we are considering a small “modeling domain” \([x,x+h]\) with \( h \) small enough that we can visualize the situation as consisting of a large number of “slices”, alternating Austenite and Martensite
phases, with \( u \) piecewise smooth. Indeed, heuristically we assume at this scale that \( u \) is piecewise linear and that the Austenite slices each have \( u_x \approx \gamma^\Lambda \) while the Martensite slices, within this domain, each have \( u_x \approx \gamma^M \), with \( \gamma^\Lambda, \gamma^M \) taken (locally) to be constants, as yet unknown. We then obtain

\[
(u(x + h, t) - u(x, t)) = \int_x^{x+h} u_x(\xi, t) \, d\xi
\]

\[
= h(\bar{z}(x, t)\gamma^M + (1 - \bar{z}(x, t))\gamma^\Lambda),
\]

where \( \bar{z} = \bar{z}(x, t) \) and \( 1 - \bar{z} \), respectively, give the local fractions of Martensite and Austenite as above. Thus, the relevant \( \gamma = \gamma(x, t) \) is effectively equal to the locally averaged value of \( u_x \) near the point \( (x, t) \), i.e.,

\[
\gamma(x, t) = \frac{\langle u_x(x, t) \rangle}{\bar{z}(x, t)\gamma^M + (1 - \bar{z}(x, t))\gamma^\Lambda}.
\]

Next, let us consider the forces across the interfaces. Assuming continuity and negligible gross motion of the interfaces relative to each other, the forces, given pointwise in view of (8) by

\[
\sigma^A(\gamma^\Lambda) = \partial_{\gamma^\Lambda} W^0(\gamma^\Lambda)
\]

and by

\[
\sigma^M(\gamma^M) = \partial_{\gamma^M} W^1(\gamma^M),
\]

must balance. This means that they must be effectively constant in the domain. Thus, for this mixture of phases to be locally stable the stress must satisfy

\[
\sigma = \partial_{\gamma^M} W^0(\gamma^\Lambda)
\]

\[
= \partial_{\gamma^M} W^1(\gamma^M), \quad \text{i.e.,} \quad \sigma^A(\gamma^\Lambda) = \sigma^M(\gamma^M)
\]

with \( [\sigma^M]'(\gamma^M) > 0 \) for stability within that phase.

Hence, we define \( \sigma \) at a point \( (x, t) \) to be this common value and wish to determine it as a function of the averaged values \( \bar{z} \) and \( \gamma \) so that we can write \( \sigma = \sigma^*(\gamma, \bar{z}) \). This just means that we obtain the stress \( \sigma^*(\gamma, \bar{z}) \) by solving (15) and (16) as a system of two equations for the two unknowns \( \gamma^M, \gamma^\Lambda \). Namely, given \( \bar{z} \) and \( \gamma \) we seek \( \gamma^M \) and \( \gamma^\Lambda \) such that

\[
\sigma^A(\gamma^\Lambda) = \sigma^M(\gamma^M),
\]

\[
\gamma = \bar{z}\gamma^M + (1 - \bar{z})\gamma^\Lambda.
\]

Upon finding the deformation gradients \( \gamma^\Lambda \) and \( \gamma^M \), we set

\[
\sigma^*(\gamma, \bar{z}) = \sigma^A(\gamma^\Lambda)(= \sigma^M(\gamma^M)).
\]

The constitutive relation (18): \( \gamma, \bar{z} \mapsto \sigma^* \) is the key to our approach.

**Remark 3.2.** Solution of system (17) for the computation of \( \sigma^* \) is not likely to be globally unique since \( \sigma^M(\cdot) \) is not injective. Indeed, this nonuniqueness is responsible for the hysteresis observed in SMAs. Since \( \sigma^A(\cdot) \), corresponding to the single-well potential \( W_0 \), is globally invertible, we note that (17) with given values of \( \bar{z} \in (0, 1) \) and \( \gamma \) is equivalent to solving

\[
- \frac{\bar{z}}{1 - \bar{z}} \gamma^M = [\sigma^A]^{-1}(\sigma^M(\gamma^M)) - \frac{1}{1 - \bar{z}} \gamma
\]

for \( \gamma^M \) and then setting \( \sigma^* = \sigma^M(\gamma^M) \). Consideration of the graph of \( \sigma^M(\cdot) \), cf. Fig. 1, shows that (19) always has a solution and, assuming, for simplicity, that the double-well potential \( W_1 \) is symmetric, that this solution will be unique with \( \gamma \) arbitrary if \( \bar{z} \) is close enough to 1 that \( \bar{z}/(1 - \bar{z}) > \max \{-[\sigma^M]'(\gamma^M)\} \) or, alternatively, will be unique with \( \bar{z} \) arbitrary in \((0, 1)\) if \( \gamma > [\sigma^A]^{-1}(\bar{z}) \) where \( \bar{z} \) is the maximum of \( \sigma^M \) between the wells. For our present purposes it would be sufficient to have local uniqueness for \( \sigma^*(\gamma, \bar{z}) \) where needed.

In this generalized setting, the partial differential equation (8) becomes

\[
u_u(x, t) = \partial_{\gamma}(\sigma^*(\gamma(x, t), \bar{z}(x, t))) + v_1 t_i
\]

with \( \gamma(x, t) = \langle u_x(x, t) \rangle \) as in (15) and with \( \bar{z} \) as our generalized control. We would expect that the locally averaged version of the solutions of (8), even with highly oscillatory gradients, should satisfy (20) in the limit as corresponding implementable oscillatory controls \( \gamma \) suitably converge to \( \bar{z} \).

Our basic assumptions are that we can consider (a sequence of) approximating control functions for which the fraction \( \bar{z} \) remains fairly constant, and that this approximate constancy remains true as the domains become small enough compared to observable regions of the wire: \( h \ll 1 \) so that we are approximating a well-defined “limit control” \( \bar{z}(x, t) \). Note that if \( \bar{z} \) is
to take values other than 0, 1 then the approximating controls \( \chi \) must rapidly alternate within these small domains in such a way that the relevant energies will remain bounded as \( \chi \to \tilde{\chi} \). If \( u_\varepsilon \), remains \( L^\infty \)-bounded during this limit pass, then (for a subsequence) we obtain weak-* convergence \( u_\varepsilon \rightharpoonup \tilde{u}_x \) as \( \chi \to \tilde{\chi} \). Hence \( \tilde{u}_x \) represents the expected value of \( u_\varepsilon \). Assuming ergodicity of the control process, we can write this expected value in the form of the local average given by (15), i.e.,

\[
\tilde{u}_x(x) = \tilde{\chi}(x,t)\gamma^M + (1 - \tilde{\chi}(x,t))\gamma^A ,
\]

where we obtain \( \tilde{\chi} \) as the measure-valued limit of \( \chi \) and then we obtain \( \gamma^M, \gamma^A \) via (17).

In view of the above modeling description we introduce the following definition.

**Definition 3.3.** Given a function \( \tilde{\chi} = \tilde{\chi}(x,t) \) taking values in \([0, 1]\) (referred to as a generalized control), we say that a solution \( u \) of (20), i.e., the global solution of the nonlinear second-order PDE

\[
u_t = \partial_x (\sigma^\varepsilon(u, \tilde{\chi}) + nu_x) 
\]

is a generalized solution of (8) corresponding to \( \tilde{\chi} \) where \( \sigma^\varepsilon \) is defined by (17)–(18).

**Remark 3.4.** We do not attempt here to obtain a direct well-posedness result, comparable to Theorem 2.1, on the existence and uniqueness of global solutions to (22). However, we do note that, under suitable hypotheses on \( \sigma^\varepsilon(\cdot, \tilde{\chi}) \), one might expect that this can be accomplished by arguments along the lines of [7].

### 4. A controllability result

In this section, we consider a situation along the lines of the previous Section 3.2. We show (locally) exact controllability in the sense of generalized solutions which are specified by Definition 3.3. For definiteness, we consider boundary conditions specifying \( u \) at \( x = 0 \) and \( u \) at \( x = 1 \).

First, we choose some \( \chi_* \) in \((0, 1)\) large enough to ensure uniqueness for the calculation of \( \sigma^\varepsilon \), cf. Remark 3.2. If we then set

\[
\gamma_* \defeq \chi_* \gamma_0 .
\]

where \( \gamma_0 > 0 \) is one of the wells of \( \mathcal{H}^\varepsilon \) where \( \sigma^M \) vanishes, we obviously have satisfied (17) with \( \gamma^A = 0 \), where \( \sigma^A \) vanishes, and \( \gamma^M = \gamma_0 \) giving \( \sigma^*(\gamma_*, \tilde{\chi}_* ) = 0 \).

We then easily see that, in the presence of the constant generalized control \( \tilde{\chi} \equiv \chi_* \), the material state given by \( U = U(x) \equiv \gamma_* x \) will be an equilibrium. This is ensured by the boundary conditions since we then have \( U_t \equiv 0 \) and \( \sigma^*(U_x, \tilde{\chi}) \equiv 0 \). Note that we also have \( U_x \equiv 0 \) so that (22) holds. Not only is \( U \) a steady state solution, but also there is no tension in the wire. We seek exact controllability for data near this state, using generalized controls uniformly near \( \chi_* \).

**Theorem 4.1.** Let \( X = \{y \in C^1(0, 1); y(0) = 0 = y'(1)\} \). Let us consider an initial state in terms of displacement and velocity, \([u_0, u_1]\), to be close to \([U, 0]\) in the sense that \( u_0 - U \) and \( u_1 \) are each small in \( X \). Let us assume that the target states \([u_0^T, u_1^T]\) are close to the state \([U, 0]\) in \( X \times X \).

Then there exists a generalized control \( \tilde{\chi} \) near \( \chi_* \), with \( \chi_* \) given by (23), and there exists a corresponding generalized solution \( u \) of (8) in the sense of Definition 3.3 attaining the target at time \( t = T \), i.e.,

\[
\begin{align*}
 u(x, T) &= u_0^T(x), & x & \in (0, 1), \\
 u_t(x, T) &= u_1^T(x), & x & \in (0, 1).
\end{align*}
\]

The generalized solution is computed with the initial and boundary conditions

\[
\begin{align*}
 u(x, 0) &= u_0(x), & x & \in (0, 1), \quad (25a) \\
 u_t(x, 0) &= u_1(x), & x & \in (0, 1), \quad (25b) \\
 u(0, t) &= 0 & for & \ t \geq 0, \quad (25c) \\
 u_x(1, t) &= \gamma_* & for & \ t \geq 0. \quad (25d)
\end{align*}
\]

**Boundary conditions** (25c) and (25d) match the material state \( U \).

**Proof.** We begin with a set of four (smooth) functions \( \varphi \) such that

\[
\begin{align*}
 \varphi_1(0) &= 1, & \varphi_1(0) &= 0, & \varphi_1(T) &= 0, \\
 \varphi'_1(T) &= 0,
\end{align*}
\]
\[ \varphi_2(0) = 0, \quad \varphi'_2(0) = 1, \quad \varphi_2(T) = 0, \]
\[ \varphi'_2(T) = 0, \]
\[ \varphi_3(0) = 0, \quad \varphi'_3(0) = 0, \quad \varphi_3(T) = 1, \]
\[ \varphi'_3(T) = 0, \]
\[ \varphi_4(0) = 0, \quad \varphi'_4(0) = 0, \quad \varphi_4(T) = 0, \]
\[ \varphi'_4(T) = 1. \] (26)

We let \( K_0, K_1, K_2 \) be maximum norm bounds on the \( \varphi_j, \varphi'_j, \) and \( \varphi''_j, \) respectively.

Using these functions, we construct \( u \) in the form
\[
\begin{align*}
  u(x,t) &= U(x) + \varphi_1(t)(u_0(x) - U(x)) + \varphi_2(t)u_1(x) \\
  &\quad + \varphi_3(t)(u_0^T(x) - U(x)) + \varphi_4(t)u_1^T(x).
\end{align*}
\] (27)

Clearly, (26) ensures that this \( u \) satisfies the initial and terminal conditions and our restriction that each data function be in \( X \) ensures that \( u \) satisfies boundary conditions (25). The argument will then be complete if we can find a generalized control function \( \tilde{\zeta} \) such that \( u \) will be a solution of Definition 3.3 we must, of course, verify that this \( \tilde{\zeta} \) is admissible (\( 0 < \tilde{\zeta} < 1 \)) at each \( x, t \) for Definition 3.3 to apply.

We first note that, with \( u \) given, we know \( u_0, \gamma = u_s, \) and \( u_{st} \) pointwise. If \( \bar{\varepsilon}_0, \bar{\varepsilon}_1 \) bound the maximum norms of each of \( u_0 - U, u_1, u_0^T - U, u_1^T \) and of their derivatives, respectively, then
\[
\begin{align*}
  |u_h| &\leq 4K_2\bar{\varepsilon}_0, \\
  |u_k - \gamma_*| &\leq 4K_0\bar{\varepsilon}_1, \\
  |u_{st}| &\leq 4K_1\bar{\varepsilon}_1. \tag{28}
\end{align*}
\]

Integrating the desired equation (22), we have
\[
\sigma|_{\zeta,t} = -\int_{\zeta}^{1} u_{tt} \, dx - vu_{st}
\]
to within an irrelevant additive constant so \( \sigma \) is known pointwise and is uniformly close to 0. We can then invert the relation: \( \sigma(\gamma, \tilde{\zeta}) = \sigma \) (with \( \gamma = u_s \) known) to obtain \( \tilde{\zeta} \) pointwise. Note that \( u_s - \gamma_* \) and \( \sigma \) are uniformly close to 0 so, e.g. by the Implicit Function Theorem, one can, indeed, solve for \( \tilde{\zeta} \), getting \( \tilde{\zeta} \) uniformly close to \( \gamma_* \) as desired. In particular, since we chose \( \chi_* \) in \( (0, 1) \) this gives the function \( \tilde{\zeta} \) so obtained also pointwise in \( (0, 1) \) provided \( \varepsilon_0, \varepsilon_1 \) are small enough so this \( \tilde{\zeta} \) is admissible for Definition 3.3 to apply. \( \square \)

**Remark 4.2.** It should be clear from the construction that we may reasonably expect to have approximate controllability using genuinely implementable controls in the setting described in the paper. With \( \tilde{\zeta} \) (and the generalized solution \( u \) already defined, we may hope to obtain approximating (highly oscillatory) controls \( \chi \) for the original problem (8) for which the discussion above correctly describes the behavior. Of course, any such function \( \tilde{\zeta} \) is obtainable as a limit of \( \{0,1\} \)-valued functions. In other words, it may be approximated by highly oscillatory controls \( \chi_n(x,t) \in \{0,1\} \) such that
\[
\tilde{\zeta}(x,t) = \lim_{n \to \infty} \frac{1}{h} \int_{x-h}^{x+h} \chi_n(\tau,t) \, d\tau. \tag{29}
\]

At present, we still have no rigorous proof even of the existence of corresponding solutions of (8), much less hard estimates permitting an argument for convergence to \( u \) of these solutions.

**References**


