3.123. Let \( \bar{X} \) denote the average thickness of 10 wafers. Then, \( E(\bar{X}) = 10 \) and \( V(\bar{X}) = 0.1 \).

a) \( P(9 < \bar{X} < 11) = P \left( \frac{9-10}{\sqrt{0.1}} < Z < \frac{11-10}{\sqrt{0.1}} \right) = P(-3.16 < Z < 3.16) = 0.9984 \).

The answer is \( 1 - 0.9984 = 0.0016 \).

b) \( P(\bar{X} > 11) = 0.01 \) and \( \sigma_{\bar{X}} = \frac{1}{\sqrt{n}} \).

Therefore, \( P(\bar{X} > 11) = P \left( Z > \frac{11-10}{\sqrt{0.1}} \right) = 0.01, \frac{11-10}{\sqrt{0.1}} = 2.33 \) and \( n = 54.43 \) which is rounded up to 6.

3.124. \( P(\bar{X} < 0.465) = P \left( \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{0.465 - 0.5}{0.05/\sqrt{49}} \right) = P(Z < -4.9) = 0 \)

3.125. \( \mu_{\bar{X}} = 755 \text{psi} \)
\( \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{35}{\sqrt{6}} = 14.29 \)

\( P(\bar{X} \geq 75.75) = P \left( \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \geq \frac{75.75 - 75.5}{14.29} \right) \)
\( = P(Z \geq 0.175) = 1 - P(Z \leq 0.175) \)
\( = 1 - 0.56945 = 0.4306 \)

3.126. Assuming a normal distribution,
\( \mu_{\bar{X}} = 2500 \)
\( \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{50}{\sqrt{5}} = 22.361 \)

\( P(2499 \leq \bar{X} \leq 2510) = P \left( \frac{2499 - 2500}{22.361} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq \frac{2510 - 2500}{22.361} \right) \)
\( = P(-0.04 \leq Z \leq 0.45) = P(Z \leq 0.45) - P(Z \leq -0.04) \)
\( = 0.673645 - 0.484037 = 0.1896 \)

3.127. \( \mu_{\bar{X}} = 8.2 \text{ minutes} \)
\( \sigma_{\bar{X}} = 1.5 \text{ minutes} \)
\( n = 49 \)
\( \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{15}{\sqrt{49}} = 2.143 \text{ (minutes)} \)
\( \mu_{\bar{X}} = \mu_{X} = 8.2 \)

Using the central limit theorem, \( \bar{X} \) is approximately normally distributed.
a) \( P(\bar{X} < 8) = P\left( Z < \frac{8 - 8.2}{0.2143} \right) = P(Z < -0.93) = 1 - 0.82381 = 0.1762 \)

b) \( P(8 < \bar{X} < 9) = P\left( \frac{8 - 8.2}{0.2143} < Z < \frac{9 - 8.2}{0.2143} \right) = P(Z < 3.73) - P(Z < -0.93) = 0.99990 - 0.17619 = 0.8237 \)

c) \( P(\bar{X} < 7.5) = P\left( Z < \frac{7.5 - 8.2}{0.2143} \right) = P(Z < -3.27) = 1 - 0.99946 = 0.0005 \)

3-128. \( n = 36 \)

\[
E(X) = \frac{1}{3} + \frac{2}{3} + \frac{3}{3} = 2
\]

\[
V(X) = (1-2)^2 \frac{1}{3} + (2-2)^2 \frac{1}{3} + (3-2)^2 \frac{1}{3} = \frac{2}{3}
\]

\[
\mu_X = 2, \sigma_X = \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}}
\]

\[
Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}
\]

Using the central limit theorem:

\[
P(2.1 < \bar{X} < 2.5) = P\left( \frac{2.1 - 2}{\sqrt{2}/3} < Z < \frac{2.5 - 2}{\sqrt{2}/3} \right)
\]

\[
= P(0.7348 < Z < 3.6742)
\]

\[
= P(Z < 3.6742) - P(Z < 0.7348) = 1 - 0.7688 = 0.2312
\]

3-129. \( X \sim N(20, 0.25) \)

a) \( \sigma_X = \frac{\sigma}{\sqrt{n}} = \frac{0.5}{\sqrt{40}} = 0.0791 \)

b) \( P(\bar{X} \geq 20.1) \)

\[
P(\bar{X} \geq 20.1) = P\left( \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \geq \frac{20.1 - 20}{0.0791} \right)
\]

\[
= P(Z \geq 1.26) = 1 - P(Z \leq 1.26)
\]

\[
= 1 - 0.89617 = 0.1038
\]

c) \( P(\bar{X} \geq 20.1) \)

\[
P(\bar{X} \geq 20.1) = P\left( \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \geq \frac{20.1 - 20}{0.5 / \sqrt{20}} \right)
\]

\[
= P(Z \geq 0.89) = 1 - P(Z \leq 0.89)
\]

\[
= 1 - 0.81327 = 0.1867
\]

d) The probability in part c) is greater than the probability in part b). This inequality occurs due to the decrease in sample size which contributes to the increase in variability.
3-130. Assume $\bar{X}$ is approximately normally distributed.

$$P(\bar{X} > 4985) = 1 - P(\bar{X} \leq 4985) = 1 - P\left( \frac{Z \leq \frac{4985 - 5000}{100}}{\sqrt{5}} \right)$$

$$= 1 - P(Z \leq -15.45) = 1 - 0 = 1$$

Supplemental Exercises

3-131. a) $P(X \leq 1.5) = \int_{0}^{1.5} e^{-x} \, dx = 0.7769$

b) $P(X < 1.5) = P(X \leq 1.5) = \int_{0}^{1.5} e^{-x} \, dx = 0.7769$

c) $P(1.5 < X < 3) = \int_{1.5}^{3} e^{-x} \, dx = 0.1733$

d) $P(X = 3) = 0$

e) $P(X > 3) = 1 - P(X \leq 3) = 1 - \int_{0}^{3} e^{-x} \, dx = 0.0498$

3-132. a) $\int_{x}^{\infty} e^{-x/2} \, dx = 0.2$

$$e^{-x/2} \bigg|_{x}^{\infty} = 0 - \left( -2e^{-x/2} \right) = 0.2$$

$$2e^{-2} = 0.2$$

$$x = 4.6052$$

b) $\int_{x}^{\infty} e^{-x/2} \, dx = 0.75$

$$e^{-x/2} \bigg|_{x}^{\infty} = 0 - \left( -2e^{-x/2} \right) = 0.75$$

$$2e^{-2} = 0.75$$

$$x = 1.9617$$

3-133. a) $P(X \leq 3) = 0.2 + 0.4 = 0.6$

b) $P(X > 3.5) = 0.4 + 0.3 + 0.1 = 0.8$

c) $P(2.7 < X < 5.1) = 0.4 + 0.3 = 0.7$

d) $E(X) = 2(0.2) + 3(0.4) + 5(0.3) + 8(0.1) = 3.9$

e) $V(X) = (2 - 3.9)^2 0.2 + (3 - 3.9)^2 0.4 + (5 - 3.9)^2 0.3 + (8 - 3.9)^2 0.1 = 3.99$
CHAPTER 4

Section 4.2

4.1. \( \bar{X}_1 = \bar{X} \left( \sum_{i=1}^{n} X_i \right) = \frac{1}{2n} \left( \sum_{i=1}^{n} X_i \right) = \frac{1}{2n} (2n \bar{x}) = \mu \)

\[ \bar{X}_2 = \frac{1}{n} \left( \frac{\sum_{i=1}^{n} X_i}{n} \right) = \frac{1}{n} \left( \frac{\sum_{i=1}^{n} X_i}{n} \right) = \frac{1}{n} (n \bar{x}) = \mu, \]

\( \bar{X}_1 \) and \( \bar{X}_2 \) are unbiased estimators of \( \mu \).

The variances are \( \text{Var}(\bar{X}_1) = \frac{\sigma^2}{2n} \) and \( \text{Var}(\bar{X}_2) = \frac{\sigma^2}{n} \); compare the MSE (variance in this case).

\[
\frac{\text{MSE}(\hat{\theta}_1)}{\text{MSE}(\hat{\theta}_2)} = \frac{\frac{\sigma^2}{2n}}{\frac{\sigma^2}{n}} = \frac{n}{2n} = \frac{1}{2}
\]

Since both estimators are unbiased, examination of the variances would conclude that \( \bar{X}_1 \) is the “better” estimator with the smaller variance.

4.2. \( \hat{\theta}_1 = \frac{1}{9} \left[ E(X_1) + E(X_2) + \cdots + E(X_6) \right] = \frac{1}{9} (9\bar{x}) = \frac{1}{9} (9\mu) = \mu \)

\( \hat{\theta}_2 = \frac{1}{2} \left[ E(3X_1) + E(X_2) + E(2X_3) \right] = \frac{1}{2} (3\mu + \mu) = \frac{1}{2} 4\mu = 2\mu \)

(\( a) \hat{\theta}_1 \) unbiased, \( \hat{\theta}_2 \) is biased so \( \hat{\theta}_1 \) is better.

b) \( \text{Var}(\hat{\theta}_1) = \frac{1}{9^2} \left( \text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_6) \right) = \frac{1}{81} (9\sigma^2) = \frac{1}{9} \sigma^2 \)

\( \text{Var}(\hat{\theta}_2) = \frac{3\text{Var}(X_1) - 2\text{Var}(X_2) + 2\text{Var}(X_3)}{2} \)

\[ = \frac{1}{2} \left( \text{Var}(3X_1) + \text{Var}(X_2) + \text{Var}(2X_3) \right) = \frac{1}{2} (9\text{Var}(X_1) + \text{Var}(X_2) + 4\text{Var}(X_3)) \]

\[ = \frac{1}{4} (9\sigma^2 + \sigma^2 + 4\sigma^2) \]

\[ = \frac{1}{4} (14\sigma^2) \]

\( \text{Var}(\hat{\theta}_2) = \frac{7\sigma^2}{2} \)

Since both estimators are unbiased, the variances can be compared to decide which is the better estimator.

The variance of \( \hat{\theta}_1 \) is smaller than that of \( \hat{\theta}_2 \). \( \hat{\theta}_1 \) is the better estimator.

4.3. Since both \( \theta_1 \) and \( \theta_2 \) are unbiased, the variances of the estimators can be examined to determine which is the “better” estimator. The variance of \( \hat{\theta}_1 \) is smaller than that of \( \hat{\theta}_2 \) thus \( \hat{\theta}_1 \) may be the better estimator.

Relative Efficiency = \( \frac{\text{MSE}(\hat{\theta}_1)}{\text{MSE}(\hat{\theta}_2)} = \frac{\text{Var}(\hat{\theta}_1)}{\text{Var}(\hat{\theta}_2)} = \frac{2}{4} = 0.5 \)
4.4. Since both estimators are unbiased:

\[ \text{Relative Efficiency} = \frac{\text{MSE}(\hat{\theta}_1)}{\text{MSE}(\hat{\theta}_2)} = \frac{\text{V}(\hat{\theta}_1)}{\text{V}(\hat{\theta}_2)} = \frac{\frac{\sigma^2}{9}}{\frac{7\sigma^2}{2}} = \frac{2}{63} \]

4.5. \[ \frac{\text{MSE}(\hat{\theta}_1)}{\text{MSE}(\hat{\theta}_2)} = \frac{\text{V}(\hat{\theta}_1)}{\text{V}(\hat{\theta}_2)} = \frac{2}{4} = 0.5 \]

4.6. \( \mathbb{E}(\hat{\theta}_1) = 0 \quad \mathbb{E}(\hat{\theta}_2) = 0/2 \)

Bias = \( \mathbb{E}(\hat{\theta}_2) - 0 \)

\[ = \frac{0}{2} = 0 \]

\[ = \frac{0}{2} \]

\( \text{V}(\hat{\theta}_1) = 10 \quad \text{V}(\hat{\theta}_2) = 4 \)

For unbiasedness, use \( \hat{\theta}_1 \) since it is the only unbiased estimator.

As for minimum variance and efficiency we have:

\[ \text{Relative Efficiency} = \frac{\left(\text{V}(\hat{\theta}_1) + \text{Bias}^2\right)}{\left(\text{V}(\hat{\theta}_2) + \text{Bias}^2\right)} \]

where, Bias for \( \hat{\theta}_1 \) is 0.

Thus,

\[ \text{Relative Efficiency} = \frac{\left(10 + 0\right)}{\left(4 + \left(0\right)^2\right)} = \frac{40}{16 + 0^2} \]

If the relative efficiency is less than or equal to 1, \( \hat{\theta}_1 \) is the better estimator.

Use \( \hat{\theta}_1 \), when \( \frac{40}{16 + 0^2} \leq 1 \)

\[ 40 \leq (16 + 0^2) \]

\[ 24 \leq 0^2 \]

\[ 0 \leq -4.899 \text{ or } 0 \geq 4.899 \]

If \(-4.899 < 0 < 4.899\) then use \( \hat{\theta}_2 \).

For unbiasedness, use \( \hat{\theta}_1 \). For efficiency, use \( \hat{\theta}_1 \) when \(-4.899 < 0 < 4.899\) and use \( \hat{\theta}_2 \) when \(-4.899 < 0 < 4.899\).

4.7. \( \mathbb{E}(\hat{\theta}_1) = 0 \quad \text{No bias} \quad \text{V}(\hat{\theta}_1) = 12 = \text{MSE}(\hat{\theta}_1) \)

\( \mathbb{E}(\hat{\theta}_2) = 0 \quad \text{No bias} \quad \text{V}(\hat{\theta}_2) = 10 = \text{MSE}(\hat{\theta}_2) \)

\( \mathbb{E}(\hat{\theta}_3) \neq 0 \quad \text{Bias} \quad \text{V}(\hat{\theta}_3) = 6 \quad \text{includes (bias)^2} \)

To compare the three estimators, calculate the relative efficiencies:

\[ \frac{\text{MSE}(\hat{\theta}_1)}{\text{MSE}(\hat{\theta}_2)} = \frac{12}{10} = 1.2 \quad \text{since rel. eff. > 1 use } \hat{\theta}_2 \text{ as the estimator for } 0 \]

\[ \frac{\text{MSE}(\hat{\theta}_1)}{\text{MSE}(\hat{\theta}_3)} = \frac{12}{6} = 2 \quad \text{since rel. eff. > 1 use } \hat{\theta}_3 \text{ as the estimator for } 0 \]

\[ \frac{\text{MSE}(\hat{\theta}_2)}{\text{MSE}(\hat{\theta}_3)} = \frac{10}{6} = 1.68 \quad \text{since rel. eff. > 1 use } \hat{\theta}_3 \text{ as the estimator for } 0 \]

Conclusion:

\( \hat{\theta}_3 \) is the most efficient estimator with bias. \( \hat{\theta}_2 \) is would be the best "unbiased" estimator.
4.8. \( n_1 = 20, \ n_2 = 10, \ n_3 = 8 \)

Show that \( S^2 \) is unbiased:

\[
E(S^2) = E\left(\frac{20S_1^2 + 10S_2^2 + 8S_3^2}{38}\right)
\]
\[
= \frac{1}{38}\left(E(20S_1^2) + E(10S_2^2) + E(8S_3^2)\right)
\]
\[
= \frac{1}{38}\left(20\sigma_1^2 + 10\sigma_2^2 + 8\sigma_3^2\right)
\]
\[
= \frac{1}{38}\left(38\sigma^2\right)
\]
\[
= \sigma^2
\]

\[ \therefore S^2 \text{ is an unbiased estimator of } \sigma^2. \]

4.9. a) Show that \( \frac{\sum_i(X_i - \bar{X})^2}{n} \) is a biased estimator of \( \sigma^2 : \)

\[
E\left(\frac{\sum_i(X_i - \bar{X})^2}{n}\right)
\]
\[
= \frac{1}{n}\sum_{i=1}^{n} E(X_i - n\bar{X})^2
\]
\[
= \frac{1}{n}\sum_{i=1}^{n} E(X_i^2) - nE(\bar{X}^2)
\]
\[
= \frac{1}{n}\sum_{i=1}^{n} (\mu^2 + \sigma^2) - n\left(\mu^2 + \frac{\sigma^2}{n}\right)
\]
\[
= \frac{1}{n}\left(n\mu^2 + n\sigma^2 - n\mu^2 - \sigma^2\right)
\]
\[
= \frac{1}{n}(n-1)n\sigma^2
\]
\[
= \sigma^2 - \frac{\sigma^2}{n}
\]

\[ \therefore \frac{\sum_i(X_i - \bar{X})^2}{n} \text{ is a biased estimator of } \sigma^2. \]

b) Bias = \[
E\left[\frac{\sum_i(X_i^2 - n\bar{X})^2}{n}\right] - \sigma^2 = \frac{\sigma^2}{n} - \sigma^2 - \frac{\sigma^2}{n} = -\frac{\sigma^2}{n}
\]

c) Bias decreases as \( n \) increases.

4.10. Show that \( \bar{X}^2 \) is a biased estimator of \( \mu. \)
Using \( E(X^2) = V(X) + [E(X)]^2 \)
\[
\begin{align*}
E(X^2) &= \frac{1}{n^2} E\left(\sum_{i=1}^n X_i^2\right) \\
&= \frac{1}{n^2} \left[ V\left(\sum_{i=1}^n X_i\right) + \left(E\left(\sum_{i=1}^n X_i\right)^2\right)\right] \\
&= \frac{1}{n^2} \left\{ n\sigma^2 + \left(\sum_{i=1}^n \mu_i\right)^2 \right\} \\
&= \frac{1}{n} \left\{ n\sigma^2 + (n\mu)^2 \right\} \\
&= \frac{1}{n} \left\{ n\sigma^2 + n^2\mu^2 \right\} \\
E(X^2) &= \frac{\sigma^2}{n} + \mu^2
\end{align*}
\]

.. $X^2$ is a biased estimator of $\mu$.

b) $\text{Bias} = E(X^2) - \mu^2 = \frac{\sigma^2}{n} + \mu^2 - \mu^2 = \frac{\sigma^2}{n}$

c) Bias decreases as $n$ increases.

Section 4.2

4.11. a) $\alpha = P(\text{reject } H_0 \text{ when } H_0 \text{ is true})$

\[
P(\bar{X} < 13.7 \text{ when } \mu = 14) = P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq \frac{13.7 - 14}{0.3/\sqrt{10}}\right)
\]

$= P(Z \leq -2.23) = 0.0129.$

The probability of rejecting the null hypothesis when it is true is 0.0129.

b) $\beta = P(\text{accept } H_0 \text{ when } \mu < 13.5 - P(\bar{X} > 13.7 \text{ when } \mu = 13.5) = P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} > \frac{13.7 - 13.5}{0.3/\sqrt{10}}\right)$

$P(Z > 1.49) = 1 - P(Z \leq 1.49) = 1 - 0.931888 = 0.0681$

The probability of accepting the null hypothesis when it is false is 0.0681.

4.12. a) $\alpha = P(\bar{X} \leq 13.7 \text{ when } \mu = 14) = P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq \frac{13.7 - 14}{0.3/\sqrt{16}}\right) = P(Z \leq -2) = 0.$

The probability of rejecting the null, when the null is true, is 0 with a sample size of 16.

b) $\beta = P(\bar{X} > 13.7 \text{ when } \mu = 13.5) = P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} > \frac{13.7 - 13.5}{0.3/\sqrt{16}}\right) = P(Z > 2.67) = 1 - P(Z \leq 2.67)$

$= 1 - 0.99621 = 0.00379.$

The probability of accepting the null hypothesis when it is false is 0.00379.

4.13. Find the boundary of the critical region if $\alpha = 0.01$:

a) $0.01 = P\left(Z \leq \frac{c - 14}{0.3/\sqrt{5}}\right)$

What $Z$ value will give a probability of 0.01? Using Table 1 in the appendix, $Z$ value is -2.33.

Thus, $\frac{c - 14}{0.3/\sqrt{5}} = -2.33, \ c = 13.687$
4.14. \[0.05 = P\left( \frac{c-14}{0.3/\sqrt{16}} \right)\]

What Z value will give a probability of 0.05? Using Table I in the appendix, Z value is -1.65. Thus, \[\frac{c-14}{0.3/\sqrt{16}} = -1.65, \ c = 13.876\]

4.15. 

a) \[\alpha = P(X \leq 98.5) + P(X > 101.5)\]
\[= P\left( \frac{X - 100}{2/\sqrt{9}} \leq \frac{98.5 - 100}{2/\sqrt{9}} \right) + P\left( \frac{X - 100}{2/\sqrt{9}} > \frac{101.5 - 100}{2/\sqrt{9}} \right)\]
\[= P(Z \leq -2.25) + P(Z > 2.25)\]
\[= P(Z \leq -2.25) + (1 - P(Z \leq 2.25))\]
\[= 0.01222 + 0.01222 = 0.0244\]

b) \[\beta = P(98.5 \leq X \leq 101.5 \text{ when } \mu = 103)\]
\[= P\left( \frac{98.5 - 103}{2/\sqrt{9}} \leq \frac{X - 103}{2/\sqrt{9}} \leq \frac{101.5 - 103}{2/\sqrt{9}} \right)\]
\[= P(-6.75 \leq Z \leq -2.25)\]
\[= P(Z \leq -2.25) - P(Z \leq -6.75)\]
\[= 0.01222 - 0 = 0.0122\]

c) \[\beta = P(98.5 \leq X \leq 101.5 \text{ when } \mu = 105)\]
\[= P\left( \frac{98.5 - 105}{2/\sqrt{9}} \leq \frac{X - 105}{2/\sqrt{9}} \leq \frac{101.5 - 105}{2/\sqrt{9}} \right)\]
\[= P(-9.75 \leq Z \leq -5.25)\]
\[= P(Z \leq -5.25) - P(Z \leq -9.75)\]
\[= 0\]

The probability of accepting the null hypothesis when it is actually false is smaller in part c since the true mean, \(\mu = 105\), is further from the acceptance region. A larger difference exists.

4.16. Use \(n = 5\), everything else held constant:

a) \[\alpha = P(X \leq 98.5) + P(X > 101.5)\]
\[= P\left( \frac{X - 100}{2/\sqrt{5}} \leq \frac{98.5 - 100}{2/\sqrt{5}} \right) + P\left( \frac{X - 100}{2/\sqrt{5}} > \frac{101.5 - 100}{2/\sqrt{5}} \right)\]
\[= P(Z \leq -1.68) + P(Z > 1.68)\]
\[= 0.093\]

b) \[\beta = P(98.5 \leq X \leq 101.5 \text{ when } \mu = 103)\]
\[= P\left( \frac{98.5 - 103}{2/\sqrt{5}} \leq \frac{X - 103}{2/\sqrt{5}} \leq \frac{101.5 - 103}{2/\sqrt{5}} \right)\]
\[= P(-5.03 \leq Z \leq -1.68)\]
\[= P(Z \leq -1.68) - P(Z \leq -5.03)\]
\[= 0.04648 - 0\]
\[= 0.04648\]

c) \[\beta = P(98.5 \leq X \leq 101.5 \text{ when } \mu = 105)\]
\[= P\left( \frac{98.5 - 105}{2/\sqrt{5}} \leq \frac{X - 105}{2/\sqrt{5}} \leq \frac{101.5 - 105}{2/\sqrt{5}} \right)\]
\[= P(-7.27 \leq Z \leq -3.91)\]
\[= P(Z \leq -3.91) - P(Z \leq -7.27)\]
\[= 0.00005 - 0\]
\[= 0.00005\]