ESTIMATING BOX-JENKINS MODELS

1. ARMA(0,0) Model

\[ y_t = \phi_0 + a_t \]

The least squares estimator of \( \phi_0 \) is the sample mean of \( y \),
\[ \hat{\phi}_0 = \frac{\sum_{t=1}^{T} y_t}{T} = \bar{y}. \]

This estimator is obtained by minimizing the least squares criterion
\[ S = \sum_{t=1}^{T} a_t^2 = \sum_{t=1}^{T} (y_t - \hat{\phi}_0)^2 \]
with respect to \( \hat{\phi}_0 \). As it turns out, \( \bar{y} \) is also the method-of-moments estimator of \( \phi_0 \) since \( E(y_t) = \phi_0 \) and the sample mean of \( y \) can be used to estimate it.

2. AR(1) Model

\[ y_t = \phi_0 + \phi_1 y_{t-1} + a_t \]

The least squares estimators of \( \phi_1 \) and \( \phi_0 \) are, respectively,
\[ \hat{\phi}_1 = \frac{\sum_{t=2}^{T} (y_{t-1} - \bar{y}_{-1})(y_t - \bar{y})}{\sum_{t=2}^{T} (y_{t-1} - \bar{y}_{-1})} \]
\[ \hat{\phi}_0 = \bar{y} - \hat{\phi}_1 \bar{y}_{-1} \]

where \( \bar{y} = \sum_{t=2}^{T} y_t / (T - 1) \) and \( \bar{y}_{-1} = \sum_{t=2}^{T} y_{t-1} / (T - 1) \). These estimators are obtained by minimizing the least squares criterion
\[ S = \sum_{t=1}^{T} a_t^2 = \sum_{t=1}^{T} (y_t - \hat{\phi}_0 - \hat{\phi}_1 y_{t-1})^2 \]
with respect to \( \hat{\phi}_0 \) and \( \hat{\phi}_1 \).

Alternatively, one could use the method-of-moments to estimate the parameters \( \phi_0 \) and \( \phi_1 \). Consider the following two moments.
\[ E(y_t) = \frac{\phi_0}{1 - \phi_1} \]  

(1)

and

\[ \text{Corr}(y_t, y_{t-1}) = \rho_1 = \phi_1 \]  

(2)

Therefore, a consistent **method-of-moments estimate** of \( \phi_1 \) is

\[ \hat{\phi}_1 = r_1, \]  

(3)

where \( r_1 \) is the first-order sample autocorrelation coefficient. From (1) we see that the sample mean of \( y, \bar{y}, \) can be used to estimate \( \phi_0/(1 - \phi_1) \). That is,

\[ \frac{\hat{\phi}_0}{1 - \hat{\phi}_1} = \bar{y}. \]  

(4)

Substituting \( \hat{\phi}_1 = r_1 \) into (4) allows us to determine an **method-of-moments estimator** of \( \phi_0 \), namely,

\[ \hat{\phi}_0 = \bar{y}(1 - r_1). \]  

(5)

Although the least squares and method-of-moments estimators of \( \phi_0 \) and \( \phi_1 \) are not the same in finite samples, they equal each other in infinite samples.

**3.MA(1) Model**

\[ y_t = \phi_0 + a_t - \theta_1 a_{t-1} \]

Unfortunately, the least squares method cannot be used to estimate \( \phi_0 \) and \( \phi_1 \) in this model since the “data” \( a_{t-1} \) is not observable. However, we can use the method of moments to estimate these parameters. Consider that

\[ E(y_t) = \phi_0 \]  

(6)

and

\[ \rho_1 = \frac{-\theta_1}{1 + \theta_1^2}. \]  

(7)

Replacing these moments with their sample estimates, we have
\[ \hat{\phi}_0 = \bar{y} \]  
\[(8)\]

and \( \hat{\theta}_1 \) so as to satisfy the moment condition

\[ r_1 = \frac{-\hat{\theta}_1}{1 + \hat{\theta}_1^2} \]  
\[(9)\]

and, at the same time, the invertibility condition \(|\hat{\theta}_1| < 1\). Again, \( r_1 \) is the first-order sample autocorrelation coefficient of the time series \( y_t \). The two roots that will satisfy (9) are

\[ \hat{\theta}_1 = \frac{-1 \pm \sqrt{1 - 4r_1^2}}{2r_1} \]  
\[(10)\]

as long as \( r_1 \leq 1/2 \). One then just chooses the root \( \hat{\theta}_1 \) that satisfies the invertibility condition.