THE TRANSFER FUNCTION MODEL

Notation

Assume, for now, that the target variable $y_i$ is stationary (i.e. has a constant mean, constant variance, and constant covariance) and that the proposed leading indicator $x_i$ is stationary as well. The Transfer Function model is described by the following three equations.

\[ y_i = \mu + \frac{\omega(B)}{\delta(B)} x_{i-b} + \epsilon_i \]  
\[ \epsilon_i = \frac{\theta(B)}{\phi(B)} a_t \]  
\[ (x_i - \mu_x) = \frac{\theta^*(B)}{\phi^*(B)} \mu_t \]

where $\mu$ is the intercept in equation (1), $\mu_x$ is the mean of $x$ and $a_t$ and $u_t$ are white noise error terms that are uncorrelated with each other at all forward and backward lags, and the "backshift" polynomials are defined as follows:

\[ \omega(B) = \omega_0 - \omega_1 B - \omega_2 B^2 - \cdots - \omega_j B^j \]  
\[ \delta(B) = 1 - \delta_1 B - \delta_2 B^2 - \cdots - \delta_j B^j \]  
\[ \theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q \]  
\[ \phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p \]  
\[ \theta^*(B) = 1 - \theta_1^* B - \theta_2^* B^2 - \cdots - \theta_q^* B^q \]  
\[ \phi^*(B) = 1 - \phi_1^* B - \phi_2^* B^2 - \cdots - \phi_p^* B^p \]

It is assumed for stationarity and invertibility purposes that the roots of the polynomials $\delta(B)$, $\theta(B)$, $\phi(B)$, $\theta^*(B)$, and $\phi^*(B)$ are outside the unit circle (i.e. the roots are greater than one in magnitude if they are real and have a modulus greater than one if they are complex). Thus, corresponding to this assumption, it is quite important that the
variables we are analyzing, \( y \) and \( x \), be in stationary form. If they are not and, say, \( \Delta y \) and \( \Delta x \) are instead stationary, then \( \Delta y \) and \( \Delta x \) should replace \( y \) and \( x \) in equations (1) and (3) above.

Equation (1) is called the **systematic dynamics equation** of the Transfer Function model because it describes the dynamic relationship between the leading indicator, \( x \), and the target variable \( y \). What is the nature of this relationship? Obviously, when \( x \) changes it does not cause a change in \( y \) until \( b \) periods later. \( b \) is called the **delay parameter** in the Transfer Function model. The nature of the numerator and denominator polynomials, \( \omega(B) \) and \( \delta(B) \) determine whether the change in \( x \) has a **finitely-lived effect** on \( y \) (after a delay of \( b \) periods) or whether the change in \( x \) has an **infinitely-lived but diminishing effect** on \( y \). (Recall that it is assumed that the roots of the autoregressive (denominator) polynomial \( \delta(B) \) are all outside of the unit circle and this guarantees that if \( \delta(B) \) is anything other than 1 (i.e. when \( s > 0 \)) then the effect must be diminishing. We will develop more of the intuition on the relationship between \( x \) and \( y \) below.

Equation (2) is called the **error dynamics equation**. \( \varepsilon \) is the unobserved error in the systematic dynamics equation. Thus, on average, the dynamic relationship that exists between \( y \) and \( x \) is described by

\[
y_t = \mu + \frac{\omega(B)}{\delta(B)} x_{t-b}.
\]

The error \( \varepsilon \) represents the approximate nature of the relationship (1). Then equation (2) says that, to the extent that the deterministic part of the systematic dynamics between \( y \) and \( x \) (the right-hand-side of the above equation) is approximate, what is left over to be explained can be modeled by a Box-Jenkins ARMA(p,q) model. Notice that if the order of the numerator polynomial, \( r \), is equal to zero, \( \omega(B) \) reduces to \( \omega_0 \). Likewise, if the order of the denominator polynomial, \( s \), is equal to zero, \( \delta(B) \) reduces to 1. Also notice that if \( r = 0 \), \( s = 0 \), and \( \omega_0 = 0 \), equations (1) and (2) imply that \( y \) follows a Box-Jenkins ARMA(p,q) model,

\[
y_t = \mu + \frac{\theta(B)}{\phi(B)} a_t = \phi_0 + \frac{\theta(B)}{\phi(B)} a_t,
\]

and \( \mu = \phi_0 \). Therefore, the Box-Jenkins model for \( y \) is just a **special case** of the Transfer Function model where the leading indicator \( x \) has no effect on the target variable \( y \). We, of course, will be very interested in distinguishing between the cases
where $x_i$ has no effect on $y_i (\omega(B)/\delta(B) = 0)$ and when $x_i$ has a systematic effect on $y_i (\omega(B)/\delta(B) \neq 0)$. In the former case, $x_i$ turns out not to be a leading indicator of $y_i$ while in the latter case $x_i$ is a viable leading indicator of $y_i$. Thus, one of the roles of the econometrician is to determine whether or not the rational polynomial $\omega(B)/\delta(B)$ is or is not equal to zero. If $\omega(B)/\delta(B)$ is equal to zero, the econometrician (as compared to the statistician who doesn't know of or use $x_i$) should discard $x_i$ as an aid in forecasting the target variable $y_i$ and try to come up with another potential leading indicator, say $z_i$, that is useful in forecasting $y_i$.

One way the econometrician can determine whether or not $x_i$ is a useful leading indicator in forecasting $y_i$ (over and above the special case Box-Jenkins model for $y_i$) is to conduct an out-of-sample forecasting experiment and see if the forecasting accuracy of the Transfer Function model using $x_i$ better than the forecasting accuracy of a simple Box-Jenkins model for $y_i$. (Recall, if $\omega(B)/\delta(B) = 0$, equations (1) and (2) specialize to the Box-Jenkins model for, namely,$$y_i = \mu + \frac{\theta(B)}{\phi(B)} a_i = \phi_0 + \frac{\theta(B)}{\phi(B)} a_i,$$where $\mu = \phi_0$, of course. We will discuss the nature of the out-of-sample forecasting experiments in more detail later.

Equation (3) is called the leading indicator Box-Jenkins equation. In the Transfer Function model the leading indicator $x_i$ is assumed to be purely exogenous in that $x_i$ affects $y_i$ but current and past values of $y_i$ do not affect $x_i$. (This is sometimes called one-way Granger Causality.) In other words, $x_i$ follows a stochastic process of its own, namely, an independent Box-Jenkins process ARMA(p*,q*). We here use p* and q* as AR and MA orders to distinguish them from the Box-Jenkins orders p and q of the error dynamics equation (2). The assumption that $x_i$ is purely exogenous is a very important assumption when adopting the Transfer Function model to characterize the relationship between the leading indicator $x_i$ and the target variable $y_i$. If this exogeneity assumption for $x_i$ is not true, we need to use some other time series model to characterize the relationship between $x_i$ and $y_i$. (One such model is called the Vector Autoregressive model, VAR for short, and given time in this class will be discussed later.)

One advantage of making a commitment to equation (3) is that, when forecasting $y_i$ more than b periods ahead, for example, $y_{T+b+1}, y_{T+b+2}, \ldots$, etc., we need future values of $x_i$, namely $x_{T+1}, x_{T+2}, \ldots$, etc. When we are forecasting, say, $y_{T+b+1}$ the Box-Jenkins model for $x_i$ (equation(3)) can be estimated and used to produce the
forecast $\hat{x}_{t+1}$, which in turn can be used in the estimated equation (1) to produce a b+1 forecast of $y_t$, namely, $\hat{y}_{t+b+1}$. Without an estimated version of equation (3) we can’t use an estimated version of equation (1) to produce forecasts beyond b periods ahead.

Thus, the Transfer Function model of equations (1), (2), and (3) are dependent on the selection of the backshift order b, the polynomial orders r, s, p, q, p*, and q* and implicitly on the orders of differencing, say d and d*, that are required to make $y_t$ and $x_t$ stationary, respectively. If a d-order difference is needed to make $y_t$ stationary and a d*-order difference is required to make $x_t$ stationary, $\Delta^d y_t$ should replace $y_t$ in equation (1) and $\Delta^{d*} x_t$ should replace $x_t$ in equation (3) above and $\mu_x$ should be changed to be $\mu_{\Delta^{d*} x}$, the mean of the d*-differenced $x_t$ series. From a notational perspective, we can represent equations (1) - (3) as TF(b, r, s, p, q, p*, q*, d, d*).

Before we go on, let’s make some concrete choices of the Transfer Function orders d, d*, b, r, s, p, q, p*, and q* so that we can more fully appreciate the nature of the Transfer Function model represented by equations (1) – (3). Let d = d* = 0 (thus $y_t$ and $x_t$ are already stationary), b=1, r=1, s=0, p=0, q=0, p*=1 and q* = 0. Also, for simplicity let’s assume that the y-intercept in equation (1) is zero ($\mu_y = 0$) and that the mean of $x_t$ is zero ($\mu_x = 0$). Then, the Transfer Function model for this specific case can be written as

$$y_t = \frac{(\omega_0 - \omega_1)B}{1} x_{t-1} + \epsilon_t \quad (1')$$

$$\epsilon_t = a_t \quad (2')$$

$$x_t = \phi x_{t-1} + u_t \quad (3')$$

In this case the systematic dynamics equation is a two-period distributed lag in $x_t$ with a one-period delay, the error of the systematic dynamics equation is white noise ($a_t$) and the purely exogenous leading indicator follows an AR(1) Box-Jenkins process.

As another illustration, let d=d*=1, b=2, r=1, s=1, p=0, q=0, p*=0, q*=1, $\mu = \mu_{\Delta x} = 0$. Then the Transfer Function model takes the specific form

$$\Delta y_t = \frac{(\omega_0 - \omega_1)B}{(1-\delta_1 B)} \Delta x_{t-2} + \epsilon_t$$

$$(1-\delta_1 B) \Delta y_t = \omega_0 \Delta x_{t-2} - \omega_1 \Delta x_{t-3} + \delta_1 \Delta y_{t-1} + \epsilon_t - \delta_1 \epsilon_{t-1}$$

$$\Delta y_t = \omega_0 \Delta x_{t-2} - \omega_1 \Delta x_{t-3} + \delta_1 \Delta y_{t-1} + \epsilon_t - \delta_1 \epsilon_{t-1} \quad (1'')$$
\[ \epsilon_t = a_t \quad (2'') \]
\[ \Delta x_t = u_t - \theta_t^* u_{t-1} \quad (3'') \]

In this case, the systematic dynamics equation consists of \( \Delta y_t \) being explained by a two-period distributed lag in \( \Delta x_t \) with a two-period delay and a one-period lag of the endogenous variable \( \Delta y_t \). In this model, not only does \( \Delta x_t \) have a two-period delay effect on \( y_t \) but last period’s change in \( y_t \) (\( \Delta y_{t-1} \)) also has an effect on this period’s change in \( y_t \). Also the error term in the systematic dynamics equation (1'') follows an MA(1) process. From equation (2'') we can see that the error term \( \epsilon_t \) is white noise, and from equation (3'') we see that the change in the leading indicator \( \Delta x_t \) follows an MA(1) process with MA(1) parameter \( \theta_t^* \).

Thus with the various choices of \( d, d^*, b, r, s, p, q, p^*, \) and \( q^* \) we can have a very sophisticated description of the relationship that exists between the target variable \( y_t \) and the proposed leading indicator \( x_t \).

**Impulse Response Function**

For the moment let us consider the deterministic form (i.e. without the error term \( \epsilon_t \)) of the systematic dynamics equation

\[ y_t = \frac{\omega(B)}{\delta(B)} x_{t-b} \]
\[ = \frac{(\omega_0 - \omega_1 B - \cdots - \omega_p B^p)}{(1 - \delta_1 B - \cdots - \delta_s B^s)} x_{t-b} \quad (4) \]

where, for simplicity, we let \( \mu = 0 \). Assuming that the roots of the polynomial \( \delta(B) \) are all outside of the unit circle, we can write (4) in the impulse response form

\[ y_t = u_0 x_{t-b} + u_1 x_{t-b-1} + u_2 x_{t-b-2} + \cdots \quad (5) \]

an infinite distributed lag in \( x_{t-b}, x_{t-b-1}, x_{t-b-2}, \cdots \). The coefficients \( u_0, u_1, u_2, \cdots \) are called the impulse response coefficients associated with the (deterministic) systematic dynamics equation (4). The interpretation of these coefficients is as follows: Consider increasing \( x \) one unit at time \( t=0 \) and in the next period \( (t=1) \) returning it to its original value. \( u_0 \) is called the impact coefficient and represents the initial impact that the one-
period, one-unit increase in $x$ has on $y$ after a delay of $b$ periods. $\nu_1$ is the **delay-1 coefficient** that represents the effect that a one-period, one-period change in $x$ has on $y$ after a delay of $b+1$ periods. $\nu_1, \nu_2, \ldots$, have similar interpretations and are called the **delay-2, delay-3**, etc. impulse response coefficients.

For example, let $b=0$, $r=0$, and $s=0$. Then the (deterministic) systematic dynamics equation (4) becomes

$$y_i = \omega_0 x_i.$$  \hspace{1cm} (4')

Furthermore, let $x_i$ be 0 for all time periods prior to and following $t = 0$, but equal to 1 at time period $t=0$. Now what impact does this type of change on $x$ have on $y_i$? Well, $y_i$ is equal to zero except at time $t=0$ and then it is equal to $\omega_0$. Therefore, the impulse response function for equation (4') is

$$\nu_j = \begin{cases} w_0 & \text{for } j = 0 \\ 0 & \text{for } j = 1, 2, \ldots \end{cases}$$

This can be plotted as

Now consider the case where $b = 0, r = 1, s = 0$. Therefore, the (deterministic) systematic dynamics equation becomes

$$y_i = w_0 x_i - w_1 x_{i-1}.$$  \hspace{1cm} (4'')
Let \( x_t \) have the one-period change of equation (6). What is the impact of this change of \( y_t \)? Well, \( y_t \) is zero except at time \( t=0 \) and then \( y_0 = w_0 \). In the following period \( y_1 = -w_1 \), Therefore, the impulse response function for (4'') is

\[
v_j = \begin{cases} 
  w_0 & \text{for } j = 0 \\
  -w_1 & \text{for } j = 1 \\
  0 & \text{otherwise}
\end{cases}
\]

That is \( v_0 = w_0 \) and \( v_1 = -w_1 \), letting \( w_i < 0 \) and \( w_0 < |w_i| \), we have the plot of the impulse responses:

![Impulse responses plot]

Then a one-period, one-unit change on \( x_t \) gives rise to an immediate change in \( y_t \) of \( w_0 \) units and then one more change in \( y_t \) of \((-w_1 - w_0)\) units \((y_0 = w_0, y_1 = -w_1)\) in period one and thereafter \( y_t \) resumes the value of 0.

Of course, if we had contemplated the functions \( y_t = w_0 x_{t-2} \) and \( y_t = w_0 x_{t-2} - w_1 x_{t-3} \). The corresponding impulse functions would have been just like the ones above but moved to the right by two periods, the amount of the new delay of \( b = 2 \) instead of \( b = 0 \). Now consider one more deterministic, systematic dynamics equation \( b = 0, r = 1, s = 1 \):

\[
y_t = w_0 x_t - w_1 x_{t-1} + \delta_1 y_{t-1}
\]

(4''')

where we assume \( 0 < \delta_1 < 1 \), and in particular, \( 0 < \delta_1 < 1 \). Again let \( x_t \) evolve as in equation (6) and let’s see what happens to \( y_t \) over time

\( y_t = 0 \) for \( t=\ldots\ldots,-3,-2,-1 \)

\( y_0 = w_0 \)
\[ y_1 = -w_1 + \delta_1 y_0 = -w_1 + \delta_1 w_0 = c, \text{ say} \]
\[ y_2 = \delta_1 y_1 = \delta_0 (-w_1 + \delta_1 w_0) = \delta_1 c \]
\[ y_3 = \delta_1 y_2 = \delta_1^2 y_1 = \delta_1^2 c \text{ etc.} \]

The impulse response function then becomes

\[
v_j = \begin{cases} 
  w_0 & \text{for } j = 0 \\
  -w_1 + \delta_1 w_0 = c, & \text{for } j = 1 \\
  \delta_1^{j+1} c, & \text{otherwise}
\end{cases}
\]

Which, plotted is

Where, for plotting purposes, we have assumed that \( c > w_0, 0 < \delta_1 < 1 \). Therefore, given the (deterministic) systematic dynamics equation of (4''') we see that a one period, one-unit change on \( x_t \) at \( t=0 \) results in \( y_t \) being \( w_0, c, \delta_1 c, \delta_1^2 c, \ldots \), in time periods \( t=0,1,2,3,\ldots \), respectively. That is, from its original “equilibrium”, \( y_0 = 0 \), the successive deviations of \( y_t \) from this original equilibrium beginning with time \( t=0 \) are units and then one more change in \( y_t \) of \( (-w_1 - w_0) \) units \( (y_0 = w_0, y_1 = -w_1) \) in period one and thereafter \( y_t \) resumes the value of \( 0, w_0, c, \delta_1 c, \delta_1^2 c, \ldots \), etc. until long enough into the future of settles back down to its original equilibrium of \( y = 0 \). Of course, if the equation (4'') we had let \( b = 2 \) instead of \( b = 0 \), we would have the same impulse responses as before but they would be delayed two periods and the impulse response graph immediately above would be shifted to the right by two periods.
We can, of course, generalize from this set of algebraic exercises. Again consider the general (deterministic) systematic dynamics equation

\[ y_t = \frac{w(B)}{f(B)} x_{t-b} = (w_0 - w_1 B - \ldots - w_r B^r) (1 - \delta_1 B - \delta_2 B^2 - \ldots - \delta_s B^s)^{-1} x_{t-b} \]  

(4)

Is there anything in general that we can say about the impulse responses associated with such an equation. The answer is yes! Given \( b=0,1,2,\ldots \) or some integer, we know that the impulse responses are zero for lags \( j=0,1,2,\ldots,b-1 \), until \( j=b \) and then \( v_j \) will equal to \( w_0 \). If \( s=0 \) and the denominator polynomial is the scalar \( \delta(B) = 1 \), then there will be \( r \) impulse responses after \( j=b \) that will be non-zero. In summary there will be \( r+1 \) non-zero impulse responses beginning with the lag \( j=b \) when the systematic dynamics follows the equation

\[ y_t = w_0 x_{t-b} - w_1 x_{t-b-1} - w_2 x_{t-b-2} - \ldots - w_r x_{t-b-r} \]  

(4''')

But now consider when \( s \neq 0 \) and is same positive integer. Then we can say that beginning with lag \( j=b \) there will be a total of \( r+1 \) irregular impulse responses before the impulse responses begin a systematic decay to zero, either exponentially or sinusoidal (we don’t know exactly until we know the signs and magnitudes of the \( \delta_1, \delta_2,\ldots,\delta_s \) coefficients), The decay of the impulse responses is guaranteed by the assumption that the roots of \( \delta(B) \), lies outside the unit circle.

In the case of the model (4'''') the impulse response function will be given by

\[ v_j = \{w_j, \quad j=0,1,2,\ldots\} \]

In the case when \( s \neq 0 \) we have

\[ y_t = w_0 x_{t-b} - w_1 x_{t-b-1} - w_2 x_{t-b-2} - \ldots - w_r x_{t-b-r} + \delta_1 y_{t-1} + \delta_2 y_{t-2} + \ldots + \delta_s y_{t-s} \]  

(4''''')

Here the impulse response function will be of the form

\[ v_j = \begin{cases} v_0, v_1,\ldots,v_r, & \text{"irregular" responses}(r+1 \text{ of them}) \\ v_{r+1}, v_{r+2},\ldots, \text{exponentially or sinusoidal declining responses beginning } v_{r+1}, \text{with and continuing.} \end{cases} \]

In summary then, we can look at the number of periods delay before the first nonzero impulse response occurs and we will be able to determine the value of \( b \). Thereafter, \( r \) will be determined by the number of “irregular” (not part of the decay)
impulse responses before the impulse responses either because all zero. Thereafter or
decay away to zero Now whether the impulse responses decay to zero or cut off to zero
determines whether or not \( s = 0 \) or \( s \neq 0 \) and there is an autoregressive part to the
(deterministic) systematic dynamics equation. If the impulse responses cut off to zero
after \( r+1 \) irregular responses the \( s \) must equal zero. Otherwise \( s=1,2,\ldots \) or same
positive integer.

As you can see, the impulse response function can help us identify the \( b, r, \) and \( s \)
orders in the systematic dynamics equation. When we add back in the error term of the
systematic dynamics equation

\[
y_t = u + \frac{w(B)}{f(B)} x_{t-b} + \varepsilon_t
\]

Then the impulse response coefficients need to be interpreted as the expected
level of \( y_t \) at various subsequent periods give a one-time, one-unit change on \( x_t \). Of
course, when the polynomials \( w(B) \) and \( \delta(B) \) are estimated from the data resulting in
\( \hat{w}(B) = \hat{w}_0 - \hat{w}_1 B - \ldots \hat{w}_r B^r \) and \( 1 - \hat{\delta}_1 B - \hat{\delta}_2 B^2 - \ldots \hat{\delta}_s B^s \) we get the estimated impulse
response polynomial

\[
\hat{v}(B) = \frac{\hat{w}(B)}{\hat{\delta}(B)} = (\hat{v}_0 + \hat{v}_1 B + \hat{v}_2 B^2 + \ldots)
\]

and the estimated impulse response function \( (\hat{v}_j, j = 0,1,2,\ldots) \) is not always as
informative as theoretical impulse response function \( (v_j, j = 0,1,2,\ldots) \)

The Cross-Correlation Function

One drawback of using the theoretical impulse response function and its empirical
counterpart, the estimated (sample) impulse response function is that the choice of the
scales of measurement of \( y_t \) and \( x_t \) (or alternatively the scales of measurement of \( \Delta^d y_t \)
and \( \Delta^d x_t \)) affects the magnitude (but not the pattern) of the impulse response
coefficients. Alternatively, we can construct a function called the cross correlation
function that mimics the delay, irregular spike, and cutting off or declining behavior of
the impulse response function yet the correlations are, by design, between \( -1 \) and \( +1 \) and
the invariant to the choice of the scales of measurement of \( y_t \) and \( x_t \).

Let’s turn to the definition of the cross correlation function. Let \( w_t \) and \( z_t \) be two
stationary time series that are potentially related to each other. Consider the following
notation: Let
\[ \gamma_{wz}(j) = E(w_t - \mu_w)(z_{t+j} - u_z) \]  
\[ j = -3, -2, -1, 0, 1, 2, 3 \]

denote the cross-correlation between \( w_t \) and \( z_t \) at lag \( j \). Notice that the lags \( j \) can be either positive or negative. For example, if \( \gamma_{wz}(j) > 0 \), then if \( w_t \) is above (below) its mean \( \mu_w \) now then, more likely than not, \( z_t \) will be above its mean two periods from now. Of course \( \rho_{wz}(j) \) is not invariant to the scales of measurement one might choose for \( w_t \) and \( z_t \) (100’s, 1000’s, 10000’s etc.) but the cross-correlation at lag \( j \) between \( w_t \) and \( z_t \) is:

\[ \rho_{wz}(j) = \frac{\gamma_{wz}(j)}{\sqrt{\text{var}(w_t)} \cdot \sqrt{\text{var}(z_t)}} \]

where \( \gamma_{zz}(j) = E(z_t - \mu_z)(z_{t+j} - u_z) \)

and

\[ \gamma_{ww}(j) = E(w_t - \mu_w)(w_{t+j} - u_w) \]

are the autocovariance functions of \( w_t \) and \( z_t \), respectively, and thus \( \gamma_{w}(0) \) and \( \gamma_{z}(0) \) are the variance \( w_t \) and \( z_t \), respectively. By construction \(-1 < \rho_{wz}(j) < 1\) and this is the case regardless of the choice of the scale of measurement of \( w_t \) and \( z_t \). For example, if \( \rho_{wz}(j) = 0.8 \) then the correlation of \( w_t \) now with \( z_t \) two periods from now is 0.8 and if, say \( w_t \) is above its mean \( \mu_w \), now then, were likely than not, \( z_t \) will be above its mean, \( \mu_z \), two periods from now.

Of course, if \( w_t \) is purely exogenous with respect to \( z_t \) the \( \rho_{wz}(j) = 0.8 \) for \( j = \ldots, -3, -2, -1 \). That is, previous deviations of \( z_t \) from its mean do not affect current and future deviations of \( w_t \) from its mean. However, if \( w_t \) does affect \( z_t \) either concurrently or in the future( as would be expected of \( w_t \) is a leading indicator of \( z_t \) ) then measure of the \( \rho_{wz}(j) \) for \( j = 0, 1, 2, \ldots \) will be one-zero.

Let us then derive the cross-correlation functions for some simple transfer function models: consider the case of

\[ \mu_x = 0, \mu = 0, b = b, r = 0, s = 0, p = 0, q = 0, p^* = 0, q^* = 0, d = 0, d^* = 0. \] We have

\[ y_t = w_0 x_{t-b} + \varepsilon_t \]  
\[ \text{where } \varepsilon_t = a_t \]
and \( x_t = u_t \) \hspace{1cm} (11)

is the white noise process.

Then
\[
\gamma_{xy}(j) = E(x_t y_{t+j}) = E[x_t (w_0 x_{t-b+j} + a_{t+j})] = w_0 E(x_t x_{t-b+j}) + E(x_t a_{t+j}) = w_0 E(x_t x_{t-b+j}) = w_0 \gamma_{xx}(j-b)
\]

Since, by assumption, \( x_t \) and \( a_{t+j} \) are uncorrelated at all leads and lags, following from
\[
E(x_t a_{t+j}) = E(\mu_t a_{t+j}) = 0 \quad \text{for all } j. \quad \text{Therefore}
\]

\[
\gamma_{xy}(j) = \begin{cases} 
  w_0 \sigma_x^2 & \text{for } j = b \\
  0 & \text{otherwise}
\end{cases}
\]

is the autocovariance function for \( x \) and \( y \) given model (9)-(11). The cross-correlation function for \( x \) and \( y \) given this model is

\[
\rho_{xy}(j) = \frac{\gamma_{xy}(j)}{\sqrt{\gamma_{xx}(0)\gamma_{yy}(0)}} = \begin{cases} 
  \frac{w_0 \sigma_x^2}{\sqrt{\sigma_x^2 \sigma_y^2}} = w_0 \frac{\sigma_x}{\sigma_y} & \text{for } j = b \\
  0 & \text{otherwise}
\end{cases}
\]

In summary, the cross-correlation function for the model (9)-(10) is plotted as
where here we have assumed that $w_0 > 0$. There is one spike, after a delay of $b$ periods, and then also, reflecting the exogeneity of $x_t$ vis-a-vis $y_t$, the negative it cuts off logs of the cross-correlation function are all zero $\rho_{xy}(j) = 0$ for $j = \ldots, -3, -2, -1$. That is the "signature" (vis-à-vis the cross-correlation function) of the model (9)-(11) where $r=0, s=0$, and $b=1$.

Now consider a second model

\[ y_t = w_0 x_{t-b} - w_1 x_{t-b-1} + \varepsilon_t \]  \hfill (12)

\[ \varepsilon_t = a_t \]  \hfill (13)

\[ x_t = \mu_t \]  \hfill (14)

\[ b = b, r = 1, s = 0, p = 0, q = 0, p^* = 0, q^* = 0, d = 0, d^* = 0 \]

The covariance function is defined to be

\[
\gamma_{xy}(j) = E(x_t y_{t+j}) = E\left[x_t (w_0 x_{t-b+j} - w_1 x_{t-b-1+j} + a_{t+j})\right] \\
= E(w_0 x_t x_{t-b+j}) + E(-w_1 x_t x_{t-b-1+j}) + E(x_t a_{t+j}) = w_0 \gamma_{xx}(j-b) - w_1 \gamma_{yx}(j-b-1)
\]

where we have $E(x_t a_{t+j}) = E(u_t a_{t+j}) = 0$ for all $j$. Then

\[
\gamma_{xy}(j) = \begin{cases} 
  w_0 \sigma_x^2 & \text{for } j = b \\
  -w_1 \sigma_x^2 & \text{for } j = b + 1 \\
  0 & \text{otherwise}
\end{cases}
\]

This translates into the cross-correlation function of

\[
\rho_{xy}(j) = \begin{cases} 
  w_0 \frac{\sigma_x}{\sigma_y} & \text{for } j = b \\
  -w_1 \frac{\sigma_x}{\sigma_y} & \text{for } j = b + 1 \\
  0 & \text{otherwise}
\end{cases}
\]
In graphical form of the cross-correlation function can be plotted as follows:

\[ \rho_{xy}(j) \]

In this graph we have assumed that \( 0 < w_0 < -w_1 \).

Finally, consider the model \( b = b, r = 1, s = 1, p = 0, q = 0, p^* = 0, q^* = 0, d = 0, d^* = 0 \)

\[ y_t = w_0 x_{t-b} - w_1 x_{t-b-1} + \delta_1 y_{t-1} + \epsilon_t \]  \hspace{1cm} (15)

\[ \epsilon_t = a_t \]  \hspace{1cm} (16)

\[ x_t = a_t \]  \hspace{1cm} (17)

\[ \gamma_{xy}(j) = E(x_t y_{t+j}) = E[x_t(w_0 x_{t-b+j} - w_1 x_{t-b-1+j} + \delta_1 y_{t-1+j} + a_{t+j})] \]
\[ = E(w_0 x_t x_{t-b+j}) + E(-w_1 x_t x_{t-b-1+j}) + \delta_1 E(x_t y_{t-1+j}) + E(x_t a_{t+j}) \]
\[ = w_0 \gamma_{xx}(j-b) - w_1 \gamma_{xx}(j-b-1) + \delta_1 \gamma_{yx}(j-1) \]

where we have \( E(x_t a_{t+j}) = E(u_t a_{t+j}) = 0 \) for all \( j \). Then for \( j < 1, \gamma(j-1) = 0 \) because the covariance between \( x_t \) and previous lags of \( y_t \), acutely, \( y_{t-1}, y_{t-2}, \ldots \), are all zero by the pure exogeneity of the leading indicator equation. In the model (15)-(17), \( \gamma_{xy}(0) = \gamma_{xy}(1) = \ldots = \gamma_{xy}(b-1) = 0 \) because \( \gamma_{xy}(j) = E(x_t y_{t+j}) = E(\mu_t y_{t+j}) = 0 \) for \( j < b \) as well. When \( j = b \) however we have

\[ \gamma_{xy}(b+1) = w_0 \gamma_{xx}(0) = w_0 \sigma_x^2. \]

Also for \( j = b+1 \) we have

\[ \gamma_{xy}(b+1) = -w_1 \gamma_{xx}(0) + \delta_1 \gamma_{xy}(b) = -w_1 \sigma_x^2 + \delta_1 w_0 \sigma_x^2 = m, \]
For $j=b+2$ we have
\[ \gamma_{xy}(b+2) = \delta_1 \gamma_{xy}(b+1) = \delta_1 m, \]

For $j=b+3$ we have
\[ \gamma_{xy}(b+3) = \delta_1 \gamma_{xy}(b+2) = \delta_1^2 m, \]

In general then our covariance function is
\[
\gamma_{xy}(j) = \begin{cases} 
  w_0 \sigma^2_x & \text{for } j = b \\
  -w_i \sigma^2_x + \delta_1 w_0 \sigma^2_x = m & \text{for } j = b + 1 \\
  \delta_1^s m & \text{for } j = b + 1 + s \text{ and } s = 1, 2, 3, 4 \ldots.
\end{cases}
\]

This implies that the correlation function for the model (15)-(17) is
\[
\rho_{xy}(j) = \begin{cases} 
  w_0 \frac{\sigma_x}{\sigma_y} & \text{for } j = b \\
  -w_i \frac{\sigma_x}{\sigma_y} + \delta_1 w_0 \frac{\sigma_x}{\sigma_y} = c & \text{for } j = b + 1 \\
  \delta_1^s c & \text{for } j = b + 1 + s, s = 1, 2
\end{cases}
\]

In graphical form the cross-correlation function can be plotted as

\[ \rho_{xy}(j) \]
where in graphing we have assumed that $0 < w_0, c > w_0 \frac{\sigma_x}{\sigma_y} > 0$ and $0 < \delta_i < 1$. This is a “signature” cross-correlation function for a transfer function model where $b=b(\text{there is a } b \text{ period delay before the “spikes” begin, then there are}(r-1) \text{ “irregular” spikes before an exponential or sinusoidal decay begins and then thereafter, instead of cutting off as when } s=0, \text{ the cross-correlation function decays away when } s=1>0.$

The deviation of the cross-correlation function of a transfer function model is somewhat more complicated when $x_i$ is not a white noise series. However, we can calculate the cross-correlation function of the “pre-whitened” $y_i$ and $x_i$ series and obtain analogous results to those we obtained before. Assuming that $y_i$ and $x_i$ are already stationary, for example, the pre-whitened series we need to cross-correlate are,

$$y_i^* = y_i \frac{\phi(B)}{\theta(B)}$$

and

$$x_i^* = x_i \frac{\phi(B)}{\theta(B)} = \frac{\theta(B) \phi(B)}{\theta(B)} \mu_i = \mu_i$$

where the pre-filter is $\phi(B) / \theta(B) = (1 - \phi_1 B - \phi_2 B^2 - \ldots \phi_p B^p) / (1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta)$. Cross-correlating the pre-filtered leading indicator (i.e the white noise errors of the leading indicator Box-Jenkins model), $x_i^* = \mu_i$, with the pre-filtered $y$ series, $y_i^*$, produces a cross-correlation function which provides a pattern that allows us to identify the delay parameter, $b$ and numerator and denominator polynomial orders, $r$ and $s$, respectively, that correspond to the transfer function for the original data $y_i$ and $x_i$.

Applying the pre-filter $\frac{\phi(B)}{\theta(B)}$ to the systematic dynamics equation (1).

Summarizing, the cross-correlation function of a transfer function model with a systematic dynamics equation of

$$y_i = \mu + x_{i-b} \frac{w(B)}{\delta(B)} + \varepsilon_i$$

should have no spikes until $j=b$, then have $r$ more “irregular” spikes (spikes not part of a systematic decaying pattern), followed by either a cutting off behavior if $s=0$, or a decaying pattern if $s=1$ (or some other positive integer) provides the pre-filter are transfer function-model
\[
\frac{y_t}{\theta(B)} \phi(B) = x_{t-b} \frac{w(B) \phi(B)}{\delta(B)} + \frac{\phi(B)}{\delta(B)} \varepsilon_t
\]

\[
y_t^+ = x_{t-b}^+ \frac{w(B)}{\delta(B)} + \varepsilon_t^+ = \frac{w(B)}{\delta(B)} \mu_{t-b}^+ + \varepsilon_t^+ ,
\]

where \( \mu = 0 \) has been conveniently (but without loss of result) imposed. Analyzing the cross-correlation function between \( y_t^+ \) and \( x_t^+ \) (i.e. \( \mu_t \)) clearly reveals the original rational (polynomial structure \( \frac{w(B)}{\delta(B)} \)) of the relationship between the original \( y_t^+ \)'s and \( x_t^+ \)'s.

**Sample Cross-correlation Function**

When we discussed the theoretical ACF and PACF functions, it was noted that we had to construct sample estimates of them before proceeding to build a Box-Jenkins model. Similarly, we need to construct a sample cross-correlation function which hopefully closely resembles the theoretical cross-correlation function before we can build a transfer function linking a leading indicator, \( x_t \), with a target variable \( y_t \), say \( c_{xx}(0) \) and \( c_{yy}(0) \). What we need are the sample variances of the estimated pre-filtered series:

\[
\hat{y}_t^+ = y_t \frac{\hat{\phi}(B)}{\hat{\theta}(B)}
\]

and

\[
\hat{x}_t^+ = x_t \frac{\hat{\phi}(B)}{\hat{\theta}(B)}
\]

where the estimated pre-filter is

\[
\frac{\hat{\phi}(B)}{\hat{\theta}(B)} = \frac{1 - \hat{\phi}_1 B - \hat{\phi}_2 B^2 - \ldots - \hat{\phi}_p B^p}{1 - \hat{\theta}_1 B - \hat{\theta}_2 B^2 - \ldots - \hat{\theta}_p B^p}
\]

and the \( \hat{\phi}_i \) and \( \hat{\theta}_i \) have been obtained by estimating an appropriate Box-Jenkins model for \( x_t \) (we are implicitly assuming in this discussion that \( x_t \) and \( y_t \) are already stationary).

A consistent estimate of the variance of \( y_t^+ \) is
where \( \overline{y}_t^+ \) = the sample mean of the \( \hat{y}_t^+ \), namely \( \sum_{t=1}^{T} \hat{y}_t^+ / T \)

A consistent estimate of the variance of \( x_t^+ \) is

\[
c_{x_t^+,x_t^+}(0) = \frac{\sum_{t=1}^{T} \hat{\mu}_t^2}{T}
\]

(19)

where \( \hat{\mu}_t \) are the Box-Jenkins white noise residuals for the leading indicator equation (3).

A consistent cross-covariance estimate at lag \( j \) is given by

\[
c_{x_t^+,y_t^+}(0) = \begin{cases} 
\frac{1}{T} \sum_{t=1}^{T-j} \hat{\mu}_t (\hat{y}_{t+j}^+ - \overline{y}_t^+) & \text{for } j = 0,1,2,3,\ldots \\
\frac{1}{T} \sum_{t=1}^{T+j} \hat{\mu}_{t+j} (\hat{y}_t^+ - \overline{y}_t^+) & \text{for } j = 0,1,2,3,\ldots 
\end{cases}
\]

(20)

Finally the sample cross-correlation function is

\[
\gamma_{x_t^+,y_t^+}(j) = \frac{c_{x_t^+,y_t^+}(j)}{\sqrt{c_{x_t^+,x_t^+}(0)c_{y_t^+,y_t^+}(0)}} \quad j = \ldots,-3,-2,-1,0,1,2,3,\ldots
\]

(21)

This is a consistent estimate of the theoretical cross correlation function

\[
\rho_{x_t^+,y_t^+}(j) = \frac{\gamma_{x_t^+,y_t^+}(j)}{\sqrt{\gamma_{x_t^+,x_t^+}(0)\gamma_{y_t^+,y_t^+}(0)}}
\]

Under the assumption that \( x_t \) and \( y_t \) are totally correlated with each other (and thus that the pre-filtered \( x^+_t \) and \( y^+_t \) are unrelated to each other), the standard error of the estimates, \( \gamma_{x_t^+,y_t^+}(j) \), is approximately (in large samples) \( \frac{1}{\sqrt{T}} \), that is,

\[
SE(\gamma_{x_t^+,y_t^+}(j)) = \frac{1}{\sqrt{T}} \quad \text{when} \quad \rho_{x_t^+,y_t^+}(j) = 0.
\]

Then if an observed sample cross-correlation
coefficient, say \( \gamma_{x,y} (j) \), is outside of the 95% confidence interval 
\((-1.96 / \sqrt{T}, 1.96 / \sqrt{T})\), one could conclude that the theoretical cross-correlation between 
\( x^+ \) and \( y^+ \) at lag \( j \), \( \rho_{x,y} (j) \), is nonzero, using the above confidence interval, 

hopefully, we can distinguish between significant spikes in the sample cross-correlation function and the “zero” values at certain lags. What may be difficult to discuss in the sample cross-correlation function—is “cutting off” behavior and “tailing off” behavior which is the distinguishing characteristic between transfer function models with \( s=0 \) (cutting off) versus \( s=1 \) (or some other positive integer) when tailing off.

Identification of Transfer Function Models

The steps for identify a TF \((b,r,s,p,q,p^*,q^*,d,d^*)\)

Model are as follows:

1. Visually inspect plots of the leading indicator \( x_t \) and target variable \( y_t \) and determine the order of differencing needed to transfer \( x_t \) (or possibly \( \log x_t \)) to stationarity difference \((d^*)\) and the order of differencing needed to transfer \( y_t \) (or possibly \( \log y_t \)) to stationarity \((d)\). The stationary form of \( x_t \) is then \( \Delta^{d^*} x_t \) (or possibly \( \Delta^{d^*} \log(x_t) \)) while the stationary form of \( y_t \) is then \( \Delta^d y_t \) (or possibly \( \Delta^d \log(y_t) \)).

2. Fit a Box-Jenkins model for the stationary form of the leading indicator, \( x_t \), namely \( \Delta^{d^*} x_t \). You will then determine the orders \( p^* \) and \( q^* \) for the ARIMA \((p^*,d^*,q^*)\) model of the leading indicator \( x_t \).

3. Given estimated Box-Jenkins model for the leading indicator \( x_t \) form the estimated pre-filtered values \((\Delta^d y_t)\) and \((\Delta^{d^*} x_t)\). Calculate the sample cross-correlation function between these estimated pre-filtered series. Use the 95% confidence interval \((-1.96 / \sqrt{T}, 1.96 / \sqrt{T})\) to determine which sample cross-correlations are significant and which ones are not. Choose \( b_t \) to be the lag at which the first significant cross-correlation occurs, then choose \( r \) based on the number of “irregular” spikes in the sample cross-correlation minus one and then choose \( s=0 \) if the sample cross-correlation function cuts off, and \( s=1 \) (or some other positive integer) if, after the irregular spikes, the sample cross-correlation function systematically tails off.
(4) Estimate the suggested transfer model using the b, r, and s values you determined in step (3). Also for your chosen value of b, estimate additional models, if any, suggested by the sample cross-correlation function. Between the competing models choose the model that has the smallest goodness-of-fit measures, AIC and SRC, white noise residuals and statistically significant coefficient (apart from possibly the $\mu$).

(5) In certain instances you may not be able to find values of r and s, for you given b value, which will produce white noise residuals. If so, you need to fit a Box-Jenkins model to the residuals $\hat{e}_t$. Obtain reasonable values for p and q for the Box-Jenkins model of the residuals of your systematic dynamics equation (1). This is called “mapping up” the auto correlation in the residuals of the systematic dynamics equation. In so doing will have tentative values of b, r, s, and p and q, that produce the smallest goodness of fit measures AIC and SBC, white residuals, and statistically significant coefficients.

(6) Before making the tentative model of step(5) the final model of choice, you need to examine the t-statistics of four overfitting models. Given a b value (the delay parameter), there is one over-fitting model for each of the dimensions, r, s, p, and q incrementing one order while holding the rest of the orders fixed at the tentative choice. If each the t-statistics of the overfitting parameters of the four overfitting models are each statistic less than 1.96 or the absolute value of the t statistic is less than 1.96 or the p-value of the t-statistic is greater than 0.05), then “fall back” to the tentative model of step (5) and make final choice.

(7) Use your transfer-function-model model in an out of sample forecasting experiment and compare the forecasting accuracy of the model (using either MAE, MSE, or the boss’s loss function of choice) with the forecasting accuracy of a properly chosen Box-Jenkins model produces more accurate forecasts (using the leading indicator) then the Box-Jenkins model (which ignores the leading indicator) then the leading indicator would appear to be useful and the transfer function model should be used for future forecasting tasks. On the other hand, if the Box-Jenkins model should prove to be more accurate than the Transfer Function model, we should drop consideration of the leading indicator $x_i$ and consider building a different transfer function using another proposed leading indicator, say, $z_i$.

Estimated transfer function model for series M data set.
$y_t = \text{sales of carpet store at time } t$
$x_t = \text{housing permits issued in the county at time } t$

stationary form of $y_t: \Delta y_t, d=1$

stationary form of $x_t: \Delta x_t, d^*=1$
Box-Jenkins model for the leading indicator $x_t$ (using the M sample dataset of obs 1-120)

$\Delta x_t =$

Therefore, $p^* = 0$, $q^* = 1$. 