Lecture 6

Mean and Variance of $\hat{\beta}_1$

Sampling Distribution of $\hat{\beta}_1$

Note that

$$\hat{\beta}_1 = \sum_{i} \frac{w_i y_i}{w_i}$$

where

$$w_i = \frac{X_i - \bar{X}}{\sum_{i} (X_i - \bar{X})^2}$$

The following properties hold for the $w_i$:

1. $\sum_{i} w_i = 0$
2. $\sum_{i} w_i x_i = 1$

Property 1: $E(\hat{\beta}_1) = \beta_1$. ($\hat{\beta}_1$ is an unbiased estimator of $\beta_1$)

Proof:

$$E(\hat{\beta}_1) = E\left(\sum_{i} \frac{w_i y_i}{w_i}\right) = \sum_{i} E\left(\frac{w_i y_i}{w_i}\right)$$

$$= \sum_{i} w_i E(y_i) = \sum_{i} w_i E(\beta_0 + \beta_1 x_i + u_i)$$
\[ \sum_{i=1}^{N} w_i (\beta_0 + \beta_1 x_i) = \beta_0 \sum_{i=1}^{N} w_i + \beta_1 \sum_{i=1}^{N} w_i x_i \]

\[ = \beta_1 \]

**Property 2:** \( \text{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum(x_i - \bar{x})^2} \)

**Proof:**

\[ \hat{\beta}_1 = \sum_{i=1}^{N} w_i y_i = \sum_{i=1}^{N} w_i (\beta_0 + \beta_1 x_i + u_i) \]

\[ = \beta_0 \sum_{i=1}^{N} w_i + \beta_1 \sum_{i=1}^{N} w_i x_i + \sum_{i=1}^{N} w_i u_i \]

\[ = \hat{\beta}_1 + \sum_{i=1}^{N} w_i u_i \]

\[ \therefore \hat{\beta}_1 - \beta_1 = \sum_{i=1}^{N} w_i u_i \]

Now

\[ \text{Var}(\hat{\beta}_1) = E(\hat{\beta}_1 - E(\hat{\beta}_1))^2 = E(\hat{\beta}_1 - \beta_1)^2 \]

\[ = E\left(\sum_{i=1}^{N} w_i u_i \right)^2 \]

\[ = E\left[ \sum_{i=1}^{N} w_i^2 u_i^2 + \sum_{i \neq j} w_i w_j u_i u_j \right] \]

\[ = \sum_{i=1}^{N} w_i^2 E(u_i^2) + \sum_{i \neq j} w_i w_j E(u_i u_j) \]

\[ = \frac{\sigma^2}{\sum(x_i - \bar{x})^2} \]
\[ \sum_{i=1}^{N} w_i^2 \sigma^2 + 0 = \sum_{i=1}^{N} w_i^2 \sigma^2 = \sigma^2 \left( \sum_{i=1}^{N} \frac{(X_i - \bar{X})}{\sum_{i} (X_i - \bar{X})^2} \right)^2 \]

\[ = \frac{\sigma^2}{\left( \sum_{i} (X_i - \bar{X})^2 \right)^2} \times \sum_{i} (X_i - \bar{X})^2 \]

\[ = \frac{\sigma^2}{\sum_{i} (X_i - \bar{X})^2} \]

**Property 3:** The sampling distribution of \( \hat{\beta}_1 \)

\[ \text{is } N \left( \beta_1, \frac{\sigma^2}{\sum_{i} (X_i - \bar{X})^2} \right) \]

**Proof:**

If SLR.6 holds \((y_i \sim N(0, \sigma^2))\) then

\[ \hat{\beta}_1 = \sum_{i=1}^{N} w_i y_i = \beta_1 + \sum_{i=1}^{N} w_i u_i \]

But \( \sum_{i=1}^{N} w_i y_i \) is a linear combination of normal random variables and by the reproductive property of the normal distribution (i.e., a linear combination of normal random variables is again a normal random variable) is also normally distributed with mean zero and

\[ \text{Var} \left( \sum_{i=1}^{N} w_i y_i \right) = \frac{\sigma^2}{\sum_{i} (X_i - \bar{X})^2} \]
Now $\hat{\beta}_1$ is equal to $\beta_1$, plus a normal random variable with mean 0 and variance $\frac{\sigma^2}{\sum (x_i - \bar{x})^2}$.

Thus $\hat{\beta}_1 \sim N(\beta_1, \frac{\sigma^2}{\sum (x_i - \bar{x})^2})$.

Thus, under SLR. 6, the sampling distribution of $\hat{\beta}_1$ is pictured below (assuming $N > 120$).

![Normal Distribution of $\hat{\beta}_1$](image)

Note: The less the variation in the $X_i$ around $\bar{X}$, the greater the sampling distribution of $\hat{\beta}_1$. The variance of the $\hat{\beta}_1$.

$$\text{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum (x_i - \bar{x})^2}.$$ If there is no variation in the $X_i$ we have $\text{Var}(\hat{\beta}_1) = \infty$. 
In a similar manner it can be shown that

Property 4: \( E(\hat{\beta}_0) = \beta_0 \) (unbiased)

Property 5: \( \text{Var}(\hat{\beta}_0) = \frac{\sum_i^{N} X_i^2}{N \sum_i^{N} (X_i - \bar{X})^2} \cdot \sigma^2 \)

Property 6: Assuming SLR.6

\[ \hat{\beta}_0 \sim N(\beta_0, \frac{\sum_i^{N} X_i^2}{N \sum_i^{N} (X_i - \bar{X})^2}) \]

Property 7: \( \text{cov}(\hat{\beta}_0, \hat{\beta}_1) = -\bar{X} \left( \frac{\sigma^2}{\sum_i^{N} (X_i - \bar{X})^2} \right) \)