Math 2343: Exam 2, Fall 2003.

Instructions:
- No notes, books or calculators.
- Do not write on the exam page.
- Clearly label your answers with the corresponding problem number.

(1) [10 pts]
The function below is the complete solution to a third-order, linear, constant-coefficient, homogeneous differential equation. Determine the ODE AND THE ICs at $t = 0$

$$y(t) = e^{2t} \sin(2t)$$

(2) [45 pts]
Solve for $y(t)$:
(a) $$y'' - 2y' + 2y = \cos t$$
(b) $$y'' - y' - 6y = 6t, \quad y(0) = \frac{1}{6}, \quad y'(0) = -1$$
(c) $$t^2 y'' - 2ty' + 3y = 0,$$

(3) [15 pts]
State the COMPLETE SOLUTION for the following differential equation but DO NOT SOLVE for any unknown coefficients:

$$y''' - 6y'' + 18y' = \cos(3t) + t^3 e^{3t} + (q_1 + q_2 t^3)e^{3t} \sin(3t).$$

(4) & (5) ON BACK.
(4) [15 pts]
You are to design a prosthetic leg out of a new composite material for a person who weighs $m = 1$. The material is elastic such that when bent it pushes back like a spring. However, it also absorbs energy to dampen the motion. Thus, prosthetic can be modeled as a spring-damper such that with the weight of the person the person-prosthetic system can be modeled as a mass-spring:

$$my'' + by' + ky = 0, \quad y(0) = y_0, \quad y'(0) = v_0.$$  

If the leg is to “springy” (too bouncy) then it is hard for the wearer to control. On the other hand, if the leg is too “mushy” (too soft, no bounce at all) then the leg feels “dead.”

(a) What is the condition on $b$ and $k$ such the leg is critically damped (i.e. in between springy and mushy)?

(b) If the damping constant is $b = 2$, what should be the spring constant so that the frequency of the response is $2$?

(c) The leg is given to another person who weighs $m = 2$ but likes a more “mushy” feel. With $b = 2$ what is the maximum value of the spring constant $k$ so that the prosthetic is overdamped?

(5) [15 pts] A harmonic vibration detector is being designed using a mass-spring system with negligible dissipation. Its motion is described by the ODE:

$$my'' + ky = f(t)$$

The external vibrations are of the form $f(t) = F \cos(\omega_f t)$.

(a) Determine the response of the system to the harmonic forcing when $\omega_f \neq \sqrt{k/m}$.

(b) Sketch the “Frequency Response Curve” and describe what it indicates about resonance.

(c) Assume the spring constant is $k = 4kg/s$. What should the mass be if the goal is to detect vibrations with frequency $f = 10$ cycles/s.