3.2 The Calculus of Inverse Functions

3.3 Derivatives of Logarithmic & Exponential Functions

1. Theorem: If \( f \) is a one-to-one differentiable function with inverse function \( f^{-1} \) and \( f'(f^{-1}(x)) \neq 0 \), then the inverse function is differentiable at \( a \) and

\[
(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}
\]

Replacing \( a \) by the general number \( x \), we have

\[
(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}
\]

If we write \( y = f^{-1}(x) \), then \( x = f(y) \)

\[
(f^{-1})'(x) = \frac{dy}{dx}
\]

\[
f'(f^{-1}(x)) = f'(y) = \frac{dx}{dy}
\]

\[
\frac{dy}{dx} = \frac{1}{(\frac{dx}{dy})}
\]

Or:

\[
y = f^{-1}(x) \iff x = f(y)
\]

\[
\frac{d}{dx} x = \frac{d}{dx} f(y) \Rightarrow 1 = \frac{df(y)}{dy} \cdot \frac{dy}{dx} \Rightarrow f'(y) \frac{dy}{dx} = 1
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)} = \left(\frac{dx}{dy}\right)
\]

2. Examples:
\[(f^{-1})'(a) = \tan \phi = \tan \left( \frac{\pi}{2} - \theta \right) = \cot \theta = \frac{1}{\tan \theta} = \frac{1}{f'(b)} \Rightarrow (f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))} \]

\[f(x) = 2x + \cos x \quad (f^{-1})'(1) = ?
\]

- \[f(0) = 1 \Rightarrow f^{-1}(1) = 0 \]

Thus:
\[ (f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{2 - \sin \alpha} = \frac{1}{2} \]

- \[y = f(x) = 2x + \cos x \]
- \[1 = f(0) \]

**Inverse:** \[ \alpha = f^{-1}(y) \]

\[ (f^{-1})'(1) = \frac{dy}{dx} \bigg|_{y=1} = \frac{dy}{dx} \bigg|_{x=0} = \frac{1}{2 - \sin \alpha} = \frac{1}{2} \]

3. **Natural logarithms and derivatives of logarithmic functions**

- **Natural logarithm:** logarithm with base e

\[ \log_e x = \ln x \]

\[ \ln x = y \quad \Rightarrow \quad e^y = x \]

\[ \ln(e^x) = x, \quad x \in \mathbb{R} \]

\[ e^{\ln x} = x, \quad x > 0 \]

\[ \ln e = 1 \]
Eg:

\[ \sqrt{\ln x} = 5, \quad x = ? \]

\[ \ln x (\log_e x) = 5 \implies x = e^5 \]

\[ \sqrt{e^{5-3x}} = 10 \]

\[ \ln e^{5-3x} = \ln 10 \implies 5-3x = \ln 10 \implies x = \frac{5 - \ln 10}{3} \]

Change of base formula:

\[ \log_a x = \frac{\ln x}{\ln a} \quad (a \neq 1, \quad a > 0) \]

Proof:

Let \( y = \log_a x \implies x = a^y \implies \ln x = \ln a^y = y \ln a \]

\[ \Rightarrow y = \frac{\ln x}{\ln a} \quad \text{ie} \quad \log_a x = \frac{\ln x}{\ln a} \]

(True \( \log_a x = \frac{\log_b x}{\log_b a}, \quad b > 0, \quad b \neq 1 \))

\[ f(x) = \log_a x \implies f'(x) = \frac{1}{x} \log_a e \]

Proof:

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\log_a (x+h) - \log_a x}{h} \]

\[ = \lim_{h \to 0} \frac{\log_a \left( \frac{x+h}{x} \right)}{h} = \lim_{h \to 0} \left[ \frac{1}{h} \log_a \left( 1 + \frac{1}{x} \right) \right] \cdot \frac{1}{x} \]

\[ = \frac{1}{x} \log_a \left( 1 + \frac{1}{x} \right) = \frac{1}{x} \log_a e \]

\[ \log_a e = \frac{\ln e}{\ln a} = \frac{1}{\ln a} \implies (\log_a x)' = \frac{1}{x \ln a} \]

\[ (\ln x)' = \frac{1}{x} \cdot \frac{1}{\ln e} = \frac{1}{x} \implies (\ln x)' = \frac{1}{x} \]