4.2 The Mean Value Theorem

Rolle's Theorem

If function \( f \): \[
\begin{align*}
1. \text{ continuous on } [a, b] \\
2. \text{ differentiable on } (a, b) \\
3. f(a) = f(b)
\end{align*}
\]
Then there is a number \( c \) in \((a, b)\) such that \( f'(c) = 0 \)

9. Prove that the equation \( x^2 + x - 1 = 0 \) has exactly one real root.

Solution: 
1st show that a root exist by using the Intermediate Theorem.
Let \( f(x) = x^2 + x - 1 \)
Then \( f(0) = -1 < 0 \) \( \quad \) \( f(1) = 1 > 0 \) \( \quad \) i.e. \( f(0) < 0 < f(1) \)
Since \( f \) is a polynomial, it is continuous, so there is a number between 0 and 1 such that \( f(c) = 0 \) \( \quad \) i.e. \( f(c) = 0 \)

2nd show the equation has no other real root by using Rolle's Theorem (argue by contradiction).
Suppose that it had two roots \( a \) and \( b \) \( \quad \) then \( f(a) = f(b) = 0 \)
Since \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\) \( \quad \) thus by Rolle's Theorem, there is a number \( c \) between \( a \) and \( b \) such that \( f'(c) = 0 \)
But \( f'(x) = 2x + 1 \) for all \( x \), \( f'(x) \) never 0.
This gives a contradiction. Therefore, the equation can't have two real roots, and it can't have more than one root.

The Mean Value Theorem

- f is continuous on [a, b]
- f is differentiable on (a, b)

Then there is a number c in (a, b) such that

\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]

or, equivalently

\[ f(b) - f(a) = f'(c)(b - a) \]

9. Suppose that \( f(0) = -3 \) and \( f'(x) \leq 5 \) for all \( x \). How large can \( f(2) \) possibly be?

Solution: We are given that \( f \) is differentiable (and therefore continuous) everywhere. We can apply the Mean Value Theorem on \([0, 2]\). There is a \( c \) such that

\[ f(2) - f(0) = f'(c)(2 - 0) \]

\[ \Rightarrow f(2) = f(0) + 2f'(c) = -3 + 2f'(c) \]

\[ f'(x) \leq 5 \text{ for all } x, \text{ so } f'(c) \leq 5 \Rightarrow 2f'(c) \leq 10 \]

Thus

\[ f(2) = -3 + 2f'(c) \leq -3 + 10 = 7 \]

That is, the largest possible value for \( f(2) \) is 7.
Theorem

If $f'(c) = 0$ for all $x$ in $(a, b)$, then $f$ is constant on $(a, b)$.

Corollary

If $f'(x) = g'(x)$ for all $x$ in $(a, b)$, then $f - g$ is constant on $(a, b)$, that is $f(x) = g(x) + C$ where $C$ is constant.

4.7 Antiderivatives

Definition

A function $F$ is called an antiderivative of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

For instance:

- Let $F(x) = \frac{1}{2} x^3$,

Let $H(x) = \frac{1}{3} x^3 + C$ (where $C$ is constant).

Theorem

If $F$ is an antiderivative of $f$ on an interval $I$, then the general antiderivative of $f$ on $I$ is $F(x) + C$ where $C$ is an arbitrary constant.

By (a) $f(x) = \sin x.$

- $F'(x) = f(x)$

$F(x) = -\cos x + C$.

(b) $f(x) = \sqrt{x}$.

$F(x) = \ln |x| + C$.

(c) $f(x) = x^n$.

$F(x) = \frac{1}{n+1} x^{n+1} + C$. 