1. One-to-one functions

Definition: A function $f$ is called a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2$$

Horizontal line test: A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Examples: one-to-one?

- Yes
- No

Note: a function passes the vertical line test.

2. Inverse functions

$$S = c \cdot t \quad (c = \text{const.})$$

- $S$: distance
- $c$: constant velocity
- $t$: time

$$S = f(t) = c \cdot t \quad \text{one-to-one}$$

- $S$ is a function of $t$
- $t$ required to reach distance $S$,

→ the inverse function of $f$.
Definition: Let \( f \) be a one-to-one function with domain \( A \) and range \( B \). Then its inverse function \( f^{-1} \) has domain \( B \) and range \( A \), and is defined by
\[
f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y
\]
for any \( y \) in \( B \).

Notes:

1. \( f^{-1}(x) \) mean the inverse function
\[
f^{-1}(x) = \left[ f(x) \right]^{-1} = \frac{1}{f(x)}
\]
2. domain of \( f^{-1} \) = range of \( f \)
range of \( f^{-1} \) = domain of \( f \)
3. \( f^{-1}(y) = x \quad \Leftrightarrow \quad f^{-1}(f(x)) = x \)
\[
y = f(x) \quad \Rightarrow \quad f(f^{-1}(y)) = x
\]
4. The letter \( x \) is traditionally used as the independent variable and \( y \) the dependent variable.
When concentrating on \( f^{-1} \) rather than \( f \), we usually reverse the roles of \( x \) and \( y \) in the definition and write
\[
y = f^{-1}(x)
\]
How to find the inverse function of a one-to-one function $f$:

**STEP 1:** Write $y = f(x)$

**STEP 2:** Solve this equation for $x$ in terms of $y$ (if possible)

**STEP 3:** To express $f^{-1}$ as a function of $x$, interchange $x$ and $y$. The resulting equation is $y = f^{-1}(x)$

**Example:**

$$f(x) = x^3 + 2$$

$$y = x^3 + 2$$

Solve $x$ in terms of $y$:

$$x^3 = y - 2 \Rightarrow x = \sqrt[3]{y - 2}$$

Interchange $x$ and $y$:

$$y = \sqrt[3]{x - 2}$$

The inverse function:

$$f^{-1}(x) = \sqrt[3]{x - 2}$$

**Note:**

- Rule of $f$: "Cube, then add 2.
- Rule of $f^{-1}$: "Subtract 2, then cube root."

$$f(f^{-1}(x)) = \left(\sqrt[3]{x - 2}\right)^3 + 2 = x - 2 + 2 = x$$

$$f^{-1}(f(x)) = \sqrt[3]{f(x) - 2}$$

$$= \sqrt[3]{x^3 - 2} = \sqrt[3]{x^3} = x$$

The graph of $f^{-1}$ is obtained by reflecting the graph of $f$ about the line $y = x$ when $x$ and $y$ are interchanged for $f^{-1}$.
3. Logarithmic functions

\[ f(x) = a^x \]

One-to-one \[ \rightarrow \] there is an inverse function \[ f^{-1} \]

\[ y = f(x) = a^x, \quad y = f^{-1}(x) = \log_a x \quad \iff \quad x = f(y) = a^y \]

logarithmic function with base \( a \)

- \( y = \log_a x \iff x = a^y \)
- \( \log_a 1 = 0 \)
- \( \log_a (a^x) = x \quad \text{for} \quad x \in \mathbb{R} \)
- \( a(\log_a x) = x \quad \text{for} \quad x > 0 \)
- \( \log_a (xy) = \log_a x + \log_a y \quad \text{for} \quad x > 0, \ y > 0 \)
- \( \log_a (x^r) = r \log_a x \quad \text{for} \quad r \in \mathbb{R} \)

Example:

\[ \log_2 80 - 2 \log_2 \sqrt{5} = \log_2 \frac{80}{\sqrt{5}^2} = \log_2 \frac{80}{5} = \log_2 16 = 4 \]