4.5 Optimization Problems

Steps:
1. Understand the problem
2. Draw a diagram and introduce notations
3. Establish relationships in the form of equations
4. Express the quantity to be optimized as a function of one variable by eliminating other variables using the equations
5. Find the absolute max. or min.

Review:
1. The Closed Interval Method
   \[ f \text{ on } [a, b] \]
   - Find values of \( f \) at critical numbers in \( (a, b) \)
   - Find \( f(a) \) & \( f(b) \)
   - Largest from 0 & 2 = absolute max.
   - Smallest from 0 & 2 = absolute min.

2. First Derivative Test for Absolute Extreme Values
   \( c \) is a critical number for a continuous \( f \) on an interval
   a) \( f'(x) > 0 \) for all \( x < c \) & \( f'(x) < 0 \) for all \( x > c \)
   \[ \Rightarrow f(c) \text{ absolute max. of } f \text{ on the interval} \]
   b) \( f'(x) < 0 \) for all \( x < c \) & \( f'(x) > 0 \) for all \( x > c \)
   \[ \Rightarrow f(c) \text{ absolute min.} \]

Examples:
1. A cylindrical can is to be made to hold 1000 cm\(^3\) of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Solution: To minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, sides).

\[ A = 2\pi r^2 + 2\pi rh \]

To eliminate \( r \) or \( h \), use the fact that the volume is given to be 1000 cm\(^3\), i.e.

\[ \pi r^2 h = 1000 \Rightarrow h = \frac{1000}{\pi r^2} \]
1. So \( A = 2\pi r^2 + 2\pi r \left( \frac{\frac{1000}{r^2}}{2} \right) = 2\pi r^2 + \frac{2000}{r}, \ r > 0 \)
which is the function to be minimized.

\[
A(r) = 2\pi r^2 + \frac{2000}{r} \implies A'(r) = 4\pi r - \frac{2000}{r^2} = 4\left(\frac{r^3 - 500}{r^2}\right)
\]

Critical numbers: \( A'(r) = 0 \implies r = 3\sqrt[3]{500} \)

\( r < \frac{3\sqrt[3]{500}}{\pi} \implies A'(r) < 0 \iff \text{absolute min at } r = \frac{3\sqrt[3]{500}}{\pi} \)

\( r > \frac{3\sqrt[3]{500}}{\pi} \implies A'(r) > 0 \) by the First Derivative Test.

When \( r = \frac{3\sqrt[3]{500}}{\pi} \), \( h = \frac{1000}{2r^2} = 2 \cdot \frac{3\sqrt[3]{500}}{2\pi} = 2 \).

(Note: Domain of \( A(r) \) is \([0, \infty)\), the Closed Interval Method doesn't apply in this case.)

2. \[
\begin{array}{c}
A \\
C \\
D
\end{array}
\]

\begin{align*}
& \text{still water} \\
& \text{run 8 km/h} \\
& \text{jet ski 6 km/h} \\
& A \to B \text{ as soon as possible where to land?}
\end{align*}

Solution: \( \text{let } |CD| = x \), then \( |DB| = 8 - x \) (run)

\[
|AD| = \sqrt{3^2 + x^2} \quad \text{(row)}
\]

\[
T = \frac{\sqrt{x^2 + 9}}{6} + \frac{8-x}{8}
\]

\[
\text{domain: } x \text{ in } [0, 8]
\]

Minimize \( T(x) \) on \([0, 8]\)

\[
T(x) = \frac{\sqrt{x^2 + 9}}{6} + 1 - \frac{x}{8} \implies T'(x) = \frac{x}{6\sqrt{x^2 + 9}} - \frac{1}{8}
\]

\[
x > 0 \implies T'(x) = 0 \implies \frac{x}{6\sqrt{x^2 + 9}} = \frac{1}{8} \implies 8x = 6\sqrt{x^2 + 9}
\]

\[
16x^2 = 9(x^2 + 9) \implies x = \frac{9}{2^{3/2}} \quad (x > 0)
\]

The only critical number in \([0, 8]\) is \( x = \frac{9}{2^{3/2}} \)

\[
T(0) = 1.5, \quad T\left(\frac{9}{2^{3/2}}\right) = \frac{\sqrt{33}}{6} \approx 1.42, \quad \frac{9}{2^{3/2}} \approx 2.58
\]

So absolute max occurs at \( x = \frac{9}{2^{3/2}} \) by the Closed Interval Method.