\( \mathcal{L} = \{ \text{set of all distribution functions over an interval } [a, \infty) \} \) where \( a \) will be usually taken to be equal to zero.

Assume that there exists a utility function \( U \) with the expected utility form. This implies that for each monetary outcome \( x \in [a, \infty) \), there is a fixed real number \( u(x) \) such that for any (lottery) distribution function \( F \) on \([a, \infty)\), we have

\[
U(F) = \int_a^\infty u(x) dF(x)
\]

which is the expectation of the random variable \( u(x) \) when \( x \) follows the probability distribution \( F \).

If in particular \( F \) is a discrete probability distribution assuming values \( \{x_1, x_2, \ldots\} \) with probabilities \( \{p_1, p_2, \ldots\} \), then

\[
U(F) = \sum_i u(x_i) p_i
\]

If \( F \) is an absolutely continuous probability distribution with density function \( f \), then
\[ U(F) = \int_a^\infty u(x)f(x)dx \]

The fixed real number \( u(x) \) is interpreted as the utility of the deterministic (sure) monetary outcome \( x \) and \( u : [a, \infty) \to \mathbb{R} \) is called the Bernoulli utility function.

Assume: Bernoulli utility function \( u : [a, \infty) \to \mathbb{R} \) is increasing and continuous.

**Risk Aversion.**

A decision maker exhibits risk aversion if for any lottery \( F \), the degenerate lottery that yields the expected amount \( \bar{x} = \int xF(x)dx \) with certainty is at least as good as the lottery \( F \) itself i.e.,

\[
u(\bar{x}) = u\left(\int x dF(x)\right)
\geq \int u(x)dF(x), \forall F.
\]
"Utility of any lottery does not exceed the utility of getting the average monetary payoff of the lottery with certainty."

If the decision maker is always indifferent between the two, we say that she is risk neutral.

The agent is strictly risk averse if

(i) he is risk averse

(ii) he is indifferent between any lottery $F$ and the certainty amount $\bar{x} = \int x F(x) dx$ only if $F$ is the degenerate distribution that assumes value $\bar{x}$ with probability one.

The inequality:

$$u\left(\int x dF(x)\right) \geq \int u(x) dF(x), \forall F.$$  

defining risk aversion is called Jensen’s inequality and holds if and only if $u$ is a concave function.
Risk aversion is equivalent to the concavity of the Bernoulli utility function $u$

(diminishing marginal utility for differentiable $u$).

Strict risk aversion:

$$u\left(\int x \, dF(x)\right) > \int u(x) \, dF(x), \forall F \text{ non-degenerate}$$

if and only if $u$ is strictly concave.

(strictly diminishing marginal utility for differentiable $u$).

Risk neutrality

$$u\left(\int x \, dF(x)\right) = \int u(x) \, dF(x), \forall F.$$ 

if and only if $u$ is linear.

Can also define risk loving behavior:

$$u\left(\int x \, dF(x)\right) \leq \int u(x) \, dF(x), \forall F.$$ 

which is equivalent to convexity of $u$. 
For a given Bernoulli utility function \( u \):

Define:

CERTAINTY EQUIVALENT of a lottery with distribution \( F \), denoted by \( c(F, u) \) by

\[
u(c(F, u)) = \int u(x) dF(x)\]
i.e., the consumer is indifferent between the (possibly uncertain) lottery \( F \) and receiving the amount of money \( c(F, u) \) with certainty.

First note that if decision maker is risk averse (\( u \) concave), then

\[
c(F, u) \leq \int x dF(x)\]
i.e., the certainty equivalent does not exceed the expected payoff from the lottery.

This follows from the fact that risk aversion implies:

\[
u\left(\int x dF(x)\right) \geq \int u(x) dF(x) = u(c(F, u))\]
so that
\[ u\left(\int x dF(x)\right) \geq u(c(F, u)) \]
and since \( u \) is increasing, we have the result.

The converse is also true.

Suppose that for all lotteries \( F \)
\[ c(F, u) \leq \int x dF(x) \]
then
\[ u(c(F, u)) \leq u\left(\int x dF(x)\right) \]
\[ \int u(x) dF(x) \leq u\left(\int x dF(x)\right) \]
so that \( u \) must be concave.

Thus, RISK AVERSION

\( \iff \) concave \( u \)
\[ \iff c(F, u) \leq \int x dF(x), \forall F. \]

Can show, Strict risk aversion \iff strictly concave \( u \)

\[ \iff c(F, u) < \int x dF(x), \forall F \text{ non-degenerate.} \]

Risk neutrality: \iff linear \( u \)

\[ \iff c(F, u) = \int x dF(x), \forall F. \]

Risk loving

\[ \iff \text{convex} \ u \]

\[ \iff c(F, u) \geq \int x dF(x), \forall F. \]

Given Bernoulli utility function \( u \), a fixed amount of money \( x \) and number \( \epsilon > 0 \), consider the lottery that assigns:

probability mass \( \frac{1}{2} \) to realization \( x + \epsilon \)
probability mass $\frac{1}{2}$ to realization $x - \epsilon$.

This is a lottery whose expected payoff is $x$.

If the agent is risk averse, then

$$u(x) \geq \frac{1}{2} u(x + \epsilon) + \frac{1}{2} u(x - \epsilon).$$

Define the probability premium $\pi(x, \epsilon, u)$ by

$$u(x) = \left(\frac{1}{2} + \pi(x, \epsilon, u)\right) u(x + \epsilon) + \left(\frac{1}{2} - \pi(x, \epsilon, u)\right) u(x - \epsilon).$$

Risk aversion $\Rightarrow \pi(x, \epsilon, u) \geq 0$.

Thus, RISK AVERSION

$$\iff \pi(x, \epsilon, u) \geq 0, \forall x, \epsilon.$$
Some simple applications.

Demand for Insurance.

Strictly risk averse agent with initial wealth $w$

Runs risk of losing $D$ with probability $\pi \in (0, 1)$.

Can buy insurance.

Each unit of insurance pays $1 if the loss occurs.

It costs $q$ to buy one unit of insurance.

Suppose agent buys $\alpha$ units of insurance.

Then his expected utility is:

$$f(\alpha) = (1 - \pi)u(w - \alpha q) + \pi u(w - D - \alpha q + \alpha)$$
Agent maximizes $f(\alpha)$ with respect to $\alpha \geq 0$.

Assume $u$ is differentiable.

Then, $u'$ is (strictly) decreasing.

FOC: Optimal $\alpha^*$ satisfies

$$-q(1 - \pi)u'(w - \alpha^* q) + \pi(1 - q)u'(w - D + \alpha^*(1 - q)) \leq 0$$

$$= 0, \text{ if } \alpha^* > 0.$$

Suppose that insurance is sold in a competitive market.

Expected Profit of a firm per unit of insurance sold at market price $q$:

$$(1 - \pi)q + \pi(q - 1)$$

which is independent of quantity sold so that in equilibrium, the market price $q$ must be such that
\[(1 - \pi)q + \pi(q - 1) = 0\]

so that

\[q = \pi.\]

This is called \textit{an actuarially fair insurance premium} (equal to expected cost to a firm of providing a unit of insurance).

If we use this in consumer’s FOC:

\[
u'(w - D + \alpha^*(1 - \pi)) \\
\leq u'(w - \alpha^*\pi) \\
= u'(w - \alpha^*\pi), \text{ if } \alpha^* > 0.
\]

Since \(w > w - D, u'(w) < u'(w - D)\), hence \(\alpha^*\) must be \(> 0\).

This implies

\[u'(w - D + \alpha^*(1 - \pi)) = u'(w - \alpha^*\pi)\]
and since $u'$ is (strictly) decreasing

$$w - D + \alpha^*(1 - \pi) = w - \alpha^*\pi$$

which yields,

$$\alpha^* = D.$$ 

Thus, with an actuarially fair insurance premium, the agent insures completely.

Intuition: With actuarially fair premium $q = \pi$, the agent’s expected final wealth for any choice of $\alpha$ is

$$\begin{align*}
(1 - \pi)(w - \alpha\pi) + \pi(w - D - \alpha\pi + \alpha)
= w - \pi D
\end{align*}$$

which is independent of choice of $\alpha$.

However, choosing $\alpha = D$, enables him to reach $w - \pi D$ with certainty and clearly is the best option for a risk averse agent.
If $q > \pi$, the agent insures incompletely.

Risk Portfolio Choice.

A risk-averse agent with differentiable Bernoulli utility $u$ divides his wealth $w > 0$ between two assets:

-a safe asset that guarantees a return of $1$ per dollar invested

- a risky asset whose return per dollar invested is given by a random variable $z$ with distribution function $F(z)$ where

$$E(z) = \int zdF(z) > 1.$$ 

Let $\alpha, \beta$ denote the amounts invested in the risky and safe assets respectively.

The agent solves:

$$\max_{\alpha, \beta \geq 0} \int u(\alpha z + \beta) dF(z)$$

s.t., $\alpha + \beta = w$. 
equivalent to
\[
\max \int u(w + \alpha(z - 1))dF(z)
\]
s.t., \(0 \leq \alpha \leq w\).

FOC: Optimal \(\alpha^*\) satisfies:
\[
\int u'(w + \alpha^*(z - 1))(z - 1)dF(z) \\
\leq 0, \text{ if } \alpha^* < w \\
= 0, \text{ if } 0 < \alpha^* < w \\
\geq 0, \text{ if } 0 < \alpha^*.
\]

Note that \(\alpha^* = 0\) cannot satisfy FOC since
\[
\int u'(w)(z - 1)dF(z) \\
= u'(w)[\{\int zdF(z)\} - 1] > 0.
\]

So, optimal portfolio has \(\alpha^* > 0\) with investment in the risky asset.

If risk is actuarially favorable, a risk averse agent will accept at least a small part of it.
Measurement of Risk Aversion.

Let $u$ be a twice differentiable concave Bernoulli utility function of an agent, $u' > 0, u'' \leq 0$.

The *Arrow-Pratt* coefficient of *absolute* risk aversion at $x$ is defined as

$$r_A(x) = -\frac{u''(x)}{u'(x)}.$$

It is a *local* measure of the curvature of $u$.

Cannot use just $u''$ as that would not be invariant to positive linear transformations of $u$.

For any $\epsilon > 0$ consider the probability premium $\pi(x, \epsilon, u)$, denoted hereafter as $\pi(\epsilon)$, defined by:

$$u(x) = \left(\frac{1}{2} + \pi(\epsilon)\right)u(x + \epsilon) + \left(\frac{1}{2} - \pi(\epsilon)\right)u(x - \epsilon)$$
Differentiating this identity twice w.r.t. $\epsilon$ and evaluating it at zero yields:

$$4\pi'(0) = -\frac{u''(x)}{u'(x)} = r_A(x)$$

so that $r_A(x)$ measures the rate at which the probability premium increases at certainty when existing wealth is $x$ and with a small additive risk of the order of $\epsilon$.

Ex. $u(x) = -\alpha \exp(-ax) + \beta$, $a > 0$, $\alpha > 0$. Then,

$$r_A(x) = a \text{ for all } x.$$ 

Constant absolute risk aversion (CARA).

Ex

$$u(x) = \frac{x^{\sigma-1}}{\sigma - 1}, \sigma \neq 1, \sigma < 2$$

$$u(x) = \ln x \text{ (if } \sigma = 1).$$

$$r_A(x) = \frac{2 - \sigma}{x}$$
Decreasing absolute risk aversion (DARA).

Given two Bernoulli utility functions $u_1$ and $u_2$ when can we say that $u_2$ exhibits (strictly) higher risk aversion globally compared to $u_1$?

(i) $r_A(x, u_2) \geq (>) r_A(x, u_1), \forall x.$

(ii) There exists an increasing, (strictly) concave function $\Psi$ such that

$$u_2(x) = \Psi(u_1(x))$$

i.e., $u_2$ is a (strictly) concave transformation of $u_1$ ($u_2$ is more concave than $u_1$).

Ex. $u(x) = x^{0.25}$ is more concave than $u(x) = x^{0.5}$.

(iii) $c(F, u_2)(<) \leq c(F, u_1), \forall F$ (non-degenerate).

(iv) $\pi(x, \epsilon, u_2) \geq (>) \pi(x, \epsilon, u_1)$ for any $x$ and $\epsilon > 0$. 
(v)

\[ \int u_2(x)dF(x) \geq u_2(\bar{x}) \]

for any (non-degenerate) \( F, \bar{x} \) implies

\[ \int u_1(x)dF(x) \geq (>)u_1(\bar{x}). \]

Any risk that would be accepted by \( u_2 \) starting from a position of certainty would also be accepted (strictly preferred) by \( u_1 \).

**Proposition:** *Definitions (i) - (v) of (strictly) more-risk-averse-than relation are equivalent.*

Note that two Bernoulli utility functions need not be comparable in terms of risk aversion - it could easily be the case that \( r_A(x, u_2) \geq r_A(x, u_1) \) at some \( x \) but \( r_A(x', u_2) < r_A(x', u_1) \) at some other \( x' \).

Application. Portfolio choice between two assets - a safe asset with return 1 for each dollar invested and a risky
asset with return given by $z$ per dollar where $\int zdF(z) > 1$.

Suppose we have two individuals with concave differentiable Bernoulli utility functions $u_1$ and $u_2$.

Let $\alpha_1^*$ and $\alpha_2^*$ denote their optimal investment in the risky asset.

Suppose $u_2$ is more risk averse than $u_1$.

We will show:

$$\alpha_2^* \leq \alpha_1^*.$$ 

FOCs:

$$\int (z - 1)u_1'(w + \alpha_1^*(z - 1))dF(z) = 0$$

$$\int (z - 1)u_2'(w + \alpha_2^*(z - 1))dF(z) = 0$$
Let $\phi(\alpha)$ be the function

$$\phi(\alpha) = \int (z - 1)u_2'(w + \alpha(z - 1))dF(z)$$

As $u_2$ is strictly concave, $\phi(\alpha)$ is strictly decreasing.

As $\phi(\alpha_2^*) = 0$, if we show that $\phi(\alpha_1^*) \leq 0$, then it follows that $\alpha_2^* \leq \alpha_1^*$.

Using definition (ii) of more risk averse,

$$u_2(x) = \Psi(u_1(x))$$

for some increasing, concave function $\Psi$.

Assume for simplicity that $\Psi$ is differentiable.

$$\phi(\alpha_1^*) = \int (z-1)\Psi'(u_1(w+\alpha_1^*(z-1)))u_1'(w+\alpha_1^*(z-1))dF(z)$$
Note that $\Psi'(u_1(w + \alpha_1^*(z - 1)))$ is a decreasing, positive function of $z$.

For $z \geq 1$, $\Psi'(u_1(w + \alpha_1^*(z - 1))) \leq \Psi'(u_1(w))$ so that

$$(z - 1)\Psi'(u_1(w + \alpha_1^*(z - 1))) \leq (z - 1)\Psi'(u_1(w))$$

For $z \leq 1$, $\Psi'(u_1(w + \alpha_1^*(z - 1))) \geq \Psi'(u_1(w))$ so that

$$(z - 1)\Psi'(u_1(w + \alpha_1^*(z - 1))) \leq (z - 1)\Psi'(u_1(w))$$

which implies that $\forall z$,

$$(z - 1)\Psi'(u_1(w + \alpha_1^*(z - 1))) \leq (z - 1)\Psi'(u_1(w))$$

and therefore,

$$\phi(\alpha_1^*)$$
$$= \int (z - 1)\Psi'(u_1(w + \alpha_1^*(z - 1)))u_1'(w + \alpha_1^*(z - 1))dF(z)$$
$$\leq \int (z - 1)\Psi'(u_1(w))u_1'(w + \alpha_1^*(z - 1))dF(z)$$
$$= \Psi'(u_1(w)) \int (z - 1)u_1'(w + \alpha_1^*(z - 1))dF(z)$$
$$= 0.$$
If we assume that $u_2$ is strictly more risk averse than $u_1$, then $\Psi$ is strictly concave in which the same arguments as above show that $\alpha_2^* < \alpha_1^*$.

Comparison across wealth levels.

"wealthier people more willing to bear risk"?

$u$ exhibits decreasing absolute risk aversion (DARA):

$$r_A(x, u)$$ is a decreasing function of $x$.

Consider two levels of wealth $x_1 > x_2$.

Consider a random risk $z$ faced by the agent (increment or decrement to wealth).

The individual evaluates the risk at wealth level $x_1$ according to the Bernoulli utility $u_1(z) = u(x_1 + z)$.

The individual evaluates the risk at wealth level $x_1$ according to the Bernoulli utility $u_2(z) = u(x_2 + z)$. 

Comparing attitude towards risk at two different wealth levels $x_1$ and $x_2$ is equivalent to comparing the utility functions $u_1$ and $u_2$.

DARA implies

$$r_A(z, u_2) = r_A(x_2 + z, u) \geq r_A(x_1 + z, u) = r_A(z, u_1), \forall z$$

**Proposition.** The following are equivalent:

(i) The Bernoulli utility function exhibits DARA

(ii) For any $x_2 < x_1$, $u(x_2 + z)$ is a concave transformation of $u(x_1 + z)$

(iii) For any risk $F(z)$, the certainty equivalent $c_x$ of the lottery formed by adding risk $z$ to wealth level $x$

$$u(c_x) = \int u(x + z) dF(z)$$

satisfies $(x - c_x)$ is decreasing in $x$. The higher wealth is, the less the agent is willing to pay to get rid of the risk.
(iv) The probability premium $\pi(x, \epsilon, u)$ is decreasing in $x$.

(v) For any $F(z)$, if

$$\int u(x_2 + z) dF(z) \geq u(x_2)$$

and $x_2 < x_1$, then

$$\int u(x_1 + z) dF(z) \geq u(x_1).$$

Relative risk aversion.

Think of risky projects whose outcomes are proportionate gains or losses of current wealth.

Let $t > 0$ denote the proportional change of wealth.

An individual with Bernoulli utility $u$ and initial wealth $x$ can evaluate a random percentage risk $t$ by the utility function

$$\tilde{u}(t) = u(tx).$$
The initial wealth corresponds to \( t = 1 \).

For small risk around \( t = 1 \), the Arrow-Pratt measure of absolute risk aversion for \( \tilde{u} \)

\[
r_A(1, \tilde{u}) = -\frac{\tilde{u}''(1)}{\tilde{u}'(1)}
\]

captures the degree of risk aversion and the latter is equal to

\[
-x\frac{u''(x)}{u'(x)}.
\]

Arrow-Pratt measure of relative risk aversion at \( x \) :

\[
r_R(x, u) = -x\frac{u''(x)}{u'(x)}.
\]

Ex.

\[
u(x) = \frac{x^{\sigma-1}}{\sigma - 1}, \sigma \neq 1, \sigma < 2
\]

\[
u(x) = \ln x \text{ (if } \sigma = 1\).
\]
\[ r_R(x) = 2 - \sigma. \]

Constant relative risk aversion.