A game is a formal representation of a situation in which individuals interact in a setting of strategic interdependence.

Strategic interdependence

⇒ each individual’s utility depends not only on his own actions but on the actions chosen by other individuals.

What action is best or "optimal" for each agent may depend on what others choose.

Therefore, decision making must take into account expectation of how other players act.
Four basic elements of a game:

*Players*

- agents that interact

*Rules*

- who moves when, what do they know or observe at each point of move, what they choose from....

*Outcomes*

- for each possible configuration of actions by all players what is the eventual outcome of the interaction - may not be quantifiable

*Payoffs*

- the players' preferences or utility function defined over possible outcomes.
Games may involve randomness (exogenous uncertainty or randomization in choice of actions)

Players may need to evaluate probability distributions or lotteries over outcomes.

Assume: each agent has preferences over all lotteries over outcomes of the game that are representable by an expected utility function.

The payoff function of a player: her Bernoulli utility: \( \{\text{space of outcomes of the game}\} \to \mathbb{R} \).

The actual utility levels are called payoffs.
Games may involve direct conflict of interest or objectives.

Ex. Matching pennies (zero sum game).
Games may involve no conflict of interest.

Ex. Pure coordination game.
Games may involve both conflict of interest and coordination problems.

Ex. Battle of Sexes.
Extensive Form Representation of a Game.

Captures

- who moves when (the sequencing of moves),

- what actions each player may choose from at each point of decision making

- what they know about other players and previous actions chosen by others at each point where they have to move in the game,

- how each configuration of action choices by players through the game generates an outcome....
Finite games: finite number of players, finite number of possible actions, finite number of moves.

Can use game tree to depict the extensive form.
Elements of a game tree:

* Decision nodes (points at which players are required to make decisions):

- Initial Nodes

- Successor Nodes

* Each action at a decision node leads to a distinct branch of the tree.

* Terminal nodes: where game terminates and an outcome of the game is realized.

* Payoff vectors at each terminal node indicating payoffs realized at that outcome.

Exogenous Uncertainty in the play of the game: modeled
as move of nature.

Games of Perfect Information:

Games where at each point of decision, every player observe all prior decisions made in course of the play of the game:

In terms of the game tree, at every decision node, players observe every action chosen in prior decision nodes that lead up to that decision node - a player knows exactly which decision node she is at.
Game of Imperfect Information.

May not observe action chosen by a previous mover in the game.

A player may not therefore know which decision node she is at.

She may know that is anywhere among a set of multiple nodes: Information Set.
In games of imperfect information, players make decisions at information sets consisting possibly of multiple nodes.

Singleton information set: Just one decision node.
Though actions chosen at an information set can lead to different outcomes depending on which node the player is really at (i.e., what unobservable actions were actually chosen in prior moves by other players)

- the player herself does not know which decision node she is at.

The set of actions she chooses from when she is at an information set must be independent of the true decision node she happens to be in.
One Shot Simultaneous Move Game: is a game of imperfect information.

No player observes the action chosen by other players when she makes her decision.
Assume: perfect recall.

Player does not forget what she observed at an earlier stage of the game.
Assume: Common knowledge of the structure of the game.
In an extensive form game, this implies all players know the extensive form.

Strategy:

A complete contingent plan or decision rule that specifies how the player will act in each possible *distinguishable* circumstance in which she might be called upon to move i.e., in each information set where she is may be possibly required to make a choice.

Given the strategies of all players, the actual play of the game may not require the players to face all contingencies that their strategy covers - all information sets may not be reached.
Definition. Let \( \mathcal{H}_i \) denote the collection of information sets where player \( i \) can possibly be required to make a decision, \( \mathcal{A} \) the set of possible actions in the game and \( C(H) \subset \mathcal{A} \) the set of actions possible at an information set \( H \).

A strategy for player \( i \) is a function \( s_i : \mathcal{H}_i \rightarrow \mathcal{A} \) such that \( s_i(H) \in C(H) \) for all \( H \in \mathcal{H}_i \).
A strategy profile in a game with $I$ players is a vector $s = (s_1, \ldots, s_I)$ where $s_i$ is the strategy chosen by player $i$.

Also denoted sometimes as $(s_i, s_{-i})$ where $s_{-i}$ is a $(I - 1)$ vector consisting of a strategy choice for each player other than player $i$. 
Normal Form Representation of a Game:

Every profile of strategies \( s = (s_1, \ldots, s_I) \) induces an outcome of the game:

- a sequence of moves actually taken

\[ \Rightarrow \text{a probability distribution over terminal nodes of the game} \]

\[ \Rightarrow \text{a probability distribution over payoff realizations of the game} \]

\[ \Rightarrow \text{expected payoff (utility) } u_i(s_1, \ldots, s_I) \text{ for each player } i. \]
Definition: For a game with $I$ players, the normal form representation $\Gamma_N$ specifies for each player $i$ a set of strategies $S_i$ (with $s_i \in S_i$) and a payoff function $u_i(s_1, \ldots, s_I)$ giving the VNM utility levels associated with the (possibly random) outcomes arising from strategies $(s_1, \ldots, s_I)$.

Formally, $\Gamma_N = [I, \{S_i\}_{i=1}^I, \{u_i(\cdot)\}_{i=1}^I]$. 
Normal form: no information about moves, order of moves, sequencing, how the "strategy" of each player is composed or played or even what it means.

Can be seen as a simultaneous move game where players choose their strategies (rather than actions at various decision nodes),
For any extensive form game, unique normal form representation.

Converse not true.
Players may randomize over actions at any decision node.

Choose probability distributions over deterministic or pure strategies.

Such randomized strategies are called mixed strategies.
Suppose that the $S_i$, the (pure) strategy set of each player $i$ is finite.

A mixed strategy by player $i$ denoted by $\sigma_i : S_i \rightarrow [0, 1]$ assigns to each pure strategy $s_i \in S_i$ a probability $\sigma_i(s_i)$ that it will be played where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.

The set of all possible mixed strategies of player $i$ is denoted by $\triangle(S_i)$. 
Every profile of mixed strategies (one for each player) generates a probability distribution over outcomes and payoffs of the game.

As players have VNM utility on the space of lotteries over outcomes, we payoff to each player from a mixed strategy profile is the expected utility (or payoff) generated.
Let $S = S_1 \times S_2 \times \ldots \times S_I$.

Let $\sigma = (\sigma_1, \ldots, \sigma_I)$ be a profile of mixed strategies where players randomize independently (not correlated strategies).

Player $i$’s VNM utility or payoff from this mixed strategy profile, denoted by $u_i(\sigma)$, is given by

$$u_i(\sigma) = \sum_{(s_1, \ldots, s_I) \in S} [\sigma_1(s_1) \ldots \sigma_I(s_I)] u_i(s_1, \ldots, s_I)$$
If strategy set is not finite, each mixed strategy is captured by a probability distribution function and the payoffs can be similarly defined.
Normal form game allowing for mixed strategies: denoted by $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i\}]$
In extensive form games, we can allow players to randomize over actions at each information set where she is required to act.

Sometimes called *behavior strategies*. 
Simultaneous Move Games (Normal Form Games).

Consider normal form game $\Gamma_N = [I, \{S_i\}_{i=1}^I, \{u_i(\cdot)\}_{i=1}^I]$ where we confine players to use only pure strategies.
Prisoner's Dilemma

\[
\begin{bmatrix}
1 & 2 & \rightarrow & \text{Not Confess} & \text{Confess} \\
\text{Not Confess} & -2, -2 & -10, -1 \\
\text{Confess} & -1, -10 & -5, -5
\end{bmatrix}
\]

(Strictly) Dominant Strategy for each player: Confess.
Let $S_{-i} = S_1 \times S_{i-1} \times S_{i+1} \ldots \times S_I$ denote the product of strategy sets of all players other than player $i$.

Definition: A strategy $s_i \in S_i$ is a strictly dominant strategy for player $i$ in a game $\Gamma_N = [I, \{S_i\}_{i=1}^I, \{u_i(.)\}_{i=1}^I]$ if for all $s_i' \neq s_i, s_i' \in S_i$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

for all $s_{-i} \in S_{-i}$.
If a player has a strictly dominant strategy, it is individually optimal for the player to play it irrespective of her belief about what other players play.

In fact, it is the unique individually optimal strategy.
If every player has a strictly dominant strategy, it is obvious that all players should play this.

However, the outcome obtained as a result may be "collectively or jointly suboptimal" or "Pareto inefficient" in the sense that all players could have been better off if they had played according to a different strategy profile.
An example of how self interested individual behavior may not be collectively good.

Reason: each player determines his or her "optimal" strategy by looking at his or her own payoff ignoring the payoffs of other players.

"Externality".
It is rare for strictly dominant strategies to exist.

What strategy is optimal for a player often depends on what other players play.

However, a rational player will never play a strategy that is dominated by some other strategy (i.e., leads to strictly lower payoff no matter what other players play).
Definition: A strategy $s_i \in S_i$ is a strictly dominated strategy for player $i$ in a game $\Gamma_N = [I, \{S_i\}_{i=1}^I, \{u_i(.)\}_{i=1}^I]$ if there exists another strategy $s'_i \in S_i$ such that

$$u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}.$$ 

In this case, we say $s'_i$ strictly dominates $s_i$. 
A strictly dominated strategy should not be played by a rational player no matter what he believes about the strategy choice of other players.

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
<tr>
<td>M</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
<tr>
<td>D</td>
<td>-2, 5</td>
<td>-3, 2</td>
</tr>
</tbody>
</table>

Both $U$ and $M$ strictly dominate $D$. 
Note that if there is a strictly dominant strategy for a player, it strictly dominates every other strategy of the player (and vice-versa).
Definition: A strategy \( s_i \in S_i \) is a weakly dominated strategy for player \( i \) in a game \( \Gamma_N = [I, \{S_i\}_{i=1}^I, \{u_i(.)\}_{i=1}^I] \) if there exists another strategy \( s_i' \in S_i \) such that

\[
u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}.
\]

and further, there exists \( \hat{s}_{-i} \in S_i \) such that

\[
u_i(s_i', \hat{s}_{-i}) > u_i(s_i, \hat{s}_{-i}).
\]

In this case, we say \( s_i' \) weakly dominates \( s_i \).
\[
\begin{bmatrix}
1 \downarrow, 2 & \rightarrow & L & R \\
U & 5,1 & 4,0 \\
M & 6,0 & 3,1 \\
D & 6,4 & 4,4 \\
\end{bmatrix}
\]

\(D\) weakly dominates \(U\) and \(M\).
If a strategy for a player weakly dominates every other strategy in the strategy set of the player, we say it is a \textit{weakly dominant} strategy.

Unlike a strictly dominated strategy, a rational player may play a weakly dominated strategy (if he/she has certain kind of belief about what the other players play).

Cannot be ruled out ex ante.
Rationality $\Rightarrow$ Rules out strictly dominated strategies.

Common knowledge of rationality

$\Rightarrow$ *Iterated Elimination of Strictly Dominated Strategies.*
Prisoner's Dilemma Modified (bias in favor of prisoner 1).

\[
\begin{bmatrix}
1 \downarrow, 2 \rightarrow & \text{Not Confess} & \text{Confess} \\
\text{Not Confess} & 0, -2 & -10, -1 \\
\text{Confess} & -1, -10 & -5, -5
\end{bmatrix}
\]
\[
1 \downarrow, 2 \rightarrow \quad L \quad M \quad R \\
T \quad -1, 7 \quad 4, 5 \quad 4, 10 \\
C \quad 0, 11 \quad 1, 4 \quad 3, 2 \\
B \quad -1, 19 \quad 2, 10 \quad 1, -1
\]
Order of deletion does not affect the set of strategies that survive iterated elimination of strictly dominated strategies.
Can generalize strictly dominated and dominant strategy concepts to normal form games that allow for mixed strategies in a straightforward way.
\[
\begin{bmatrix}
1 \downarrow, 2 & \rightarrow & L & R \\
U & 10, 1 & 0, 4 \\
M & 4, 2 & 4, 3 \\
D & 0, 5 & 10, 2
\end{bmatrix}
\]

Playing \(U\) and \(D\) with probability \(\frac{1}{2}\) each strictly dominates \(M\).
Nash Equilibrium.

Consider normal form game $\Gamma_N = [I, \{S_i\}, \{u_i\}]$ where players restrict themselves to pure strategies.

**Definition 1** A strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_I^*) \in S$ constitutes a Nash Equilibrium (NE) if for every $i = 1, \ldots, I$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

for all $s_i \in S_i$. 
Each player’s strategy is a best response to the strategies actually played by rivals.

\[ s_i^* \in b_i(s_{-i}^*), \quad i = 1, \ldots, I \]

where \( b_i(s_{-i}^*) \) is the best-response (or best reply or "reaction") correspondence defined by

\[ b_i(s_{-i}^*) = \left\{ s_i \in S_i : s_i \text{ solves } \max_{s_i \in S_i} u_i(s'_i, s_{-i}^*) \right\}. \]
* No player has a (strict) incentive to *unilaterally* deviate from playing according to strategy profile $s^*$ (does not rule out gainful deviation by a coalition of multiple players).
In a NE, players play rationally holding correct conjectures (or forecasts) of rivals’ play.

Example:

\[
\begin{bmatrix}
  b_1 & b_2 & b_3 \\
  a_1 & 0,7 & 2,5 & 7,0 \\
  a_2 & 5,2 & 3,3 & 5,2 \\
  a_3 & 7,0 & 2,5 & 0,7 \\
\end{bmatrix}
\]
Unique NE: \((a_2, b_2)\).
* Let $N$ denote the set of NE strategy profiles,

$IED$ the set of strategy profiles that survive iterated elimination of strictly dominated strategies and

$U$ the set of strategy profiles consisting of strategies that are strictly undominated.

Then,

$$N \subset IED \subset U.$$
The concept of NE is based on the concept of mutually correct expectations.

Quite often, there can be multiple NE.

Coordination problems.
Example: Coordination game.

\[
\begin{bmatrix}
L & R \\
U & 100, 100 & 0, 0 \\
D & 0, 0 & 1000, 1000
\end{bmatrix}
\]

The two NE are Pareto-ranked (both players better off in \((D, R)\) compared to \((U, L)\).
Example: (Pure coordination game)

\[
\begin{bmatrix}
L & R \\
U & 100, 100 & 0, 0 \\
D & 0, 0 & 100, 100
\end{bmatrix}
\]
Example: Battle of Sexes

\[
\begin{bmatrix}
\text{Opera} & \text{Game} \\
\text{Opera} & 100, 1000 & 0, 50 \\
\text{Game} & 50, 0 & 1000, 100
\end{bmatrix}
\]
Example: Cake eating.

A cake is to be divided among two players.

Players 1 and 2 simultaneously choose the shares \((s_1, s_2), 0 \leq s_i \leq 1\), of the cake they demand.

The payoff of each player \(i\) is the share of the cake obtained be her and is given by:

\[
x_i = \begin{cases} 
  s_i, & \text{if } s_i + s_j \leq 1, \\
  0, & \text{if } s_i + s_j > 1.
\end{cases}
\]
Set of NE = \{(s_1, s_2) : s_1 + s_2 = 1, 0 \leq s_i \leq 1, i = 1, 2\}

Continuum of NE. Conflict of objectives across NE.
Why should we expect conjectures to be correct?

Certainly not a necessary consequence of rationality or common knowledge of rationality and payoffs.
* If there is a unique predicted outcome for a game (a unique obvious way to play the game), then it must be a Nash equilibrium.
* If certain outcomes are *focal* (Schelling) for cultural or other reasons (having to do with information not contained within the description of the game), then such an outcome can be a prediction only if it is Nash equilibrium.
* If players make a *non-binding* agreement prior to play about how they are going to play the game, then such an agreement is credible only if it is a Nash equilibrium (the pre-game communication makes the agreement focal).
*Stable social convention (norm): If the game is played repeatedly, then some stable social convention about how to play the game may emerge (a limit of some dynamic adjustment process); such a stable social convention or norm must be a NE.
Mixed Strategy Nash Equilibrium.

Consider the normal form game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i\}]$

**Definition 2** A strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_I^*) \in \prod_{i=1}^{I} \Delta(S_i)$ constitutes a Nash Equilibrium (NE) if for every $i = 1, \ldots, I$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

for all $\sigma_i \in \Delta(S_i)$. 
Example (Matching Pennies)

\[
\begin{bmatrix}
H & T \\
H & +1, -1 & -1, +1 \\
T & -1, +1 & +1, -1
\end{bmatrix}
\]

There is no NE in pure strategies.

Each player playing \( H \) and \( T \) with probability \( \frac{1}{2} \) each constitutes a mixed strategy NE.

Given this strategy of rival, each player indifferent between playing \( H \) or \( T \).
In any mixed strategy NE, each player is indifferent between pure strategies that she plays with strictly positive probability.

i.e., given the mixed strategies played by other players, all such pure strategies must yield her exactly her the same expected utility or payoff (which would also be her NE payoff).
Further, no pure strategy that is played with probability zero by a player can yield strictly higher payoff than the payoff from the pure strategies that are played with strictly positive probability.

[In case the strategy set is not finite, the above must be true for almost every strategy in the support of the mixed strategy of each player].
The following proposition is written for the case of finite strategy sets and shows that the above is both necessary as well as sufficient for a mixed strategy NE:

**Proposition.** Assume $S_i$ is finite. Let $S_i^+ \subset S_i$ denote the set of pure strategies that player $i$ plays with strictly positive probability in a mixed strategy profile

$$
\sigma^* = (\sigma_1^*, ..., \sigma_I^*) \in \prod_{i=1}^I \Delta(S_i). \text{ Strategy profile } \sigma^* \text{ is a NE if and only if for all } i = 1, ... I
$$

$$(i) \ u_i(s_i, \sigma_{-i}^*) = u_i(s_i', \sigma_{-i}^*), \forall s_i, s_i' \in S_i^+
$$

$$(ii) \ u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*), \forall s_i \in S_i^+ \text{ and } \forall s_i' \in S_i - S_i^+.
$$

So to test whether a given mixed strategy profile is a NE we only need to test that all pure strategies played with strictly positive probability yield equal payoffs for each player and that no player can do better by playing some other pure strategy.
Example:

\[
\begin{bmatrix}
L & R \\
U & 100, 100 & 0, 0 \\
D & 0, 0 & 1000, 1000
\end{bmatrix}
\]

Suppose player 1 plays $U$ and $D$ with probability $p$ and $1 - p$, respectively.

For player 2, playing $L$ yields expected payoff $100p$ and playing $R$ yields $1000(1 - p)$. These two expected payoffs are equal only if $p = \frac{1}{11}$. By a symmetric argument, player 1 is indifferent between $U$ and $D$ if and only if player 2 plays $L$ and $R$ with probabilities $\frac{1}{11}$ and $\frac{10}{11}$, respectively.

Thus, these mixed strategies constitute a NE.
Dynamic Games.

Most economic interactions:

agents choose actions over time where some information over actions chosen in previous periods is available.

If previous actions unobserved or unknown to players that move later, the game is effectively a simultaneous move game.

Otherwise, its a true dynamic game.
Extensive forms capture the dynamic structure of moves.

Every extensive form game can be reduced to a normal form game where players simultaneously choose *strategies*.

Why not just use the theory of games developed for normal form games - for example, the concept of NE and apply it to the reduced normal form associated with the extensive form of a dynamic game?
Problem: Some NE strategies may not be *credible* under the sequential structure captured by the extensive form (information suppressed in the normal form).

Need refinement of Nash equilibrium.
Dynamic Finite Games of Complete and Perfect Information

Players not only have common knowledge of payoffs, but also know every choice made by players whose moves precede them.

Every information set consists of a single node.

Finite number of nodes.
Example.

- Maria

\[
\begin{bmatrix}
100 \\
1000
\end{bmatrix}
\]

- Dwain

\[
\begin{bmatrix}
1000 & 25 \\
100 & 0
\end{bmatrix}
\]

Strategy Sets:

Maria: \{Y,N\}

Dwain: \{L if N, D if N\}
Normal form:

\[
\begin{bmatrix}
  \text{L if N} & \text{D if N} \\
  100, 1000 & 100, 1000 \\
  1000, 100 & 25, 0
\end{bmatrix}
\]
Two pure strategy NE:

NE1: (N, L if N)

NE2: (Y, D if N).
NE2 is not credible because if we look at the extensive form we immediately know that Dwain would never choose D if Maria chose N.

NE2: based on Dwain playing a strategy where he threatens to play an action (D) that he would never play if he actually had to choose an action in the real play of the game - his strategy is a bluff.
The reason why the outcome in NE2 remains a Nash equilibrium in the normal form is because in the actual play of this NE, Dwain will never have to actually choose between L and D - what he says he will do in that node does not affect his payoff - his bluff will never be called.

The normal form of the game does not allow us to see this credibility problem (the strategies can be re-labelled as A,B,C, D and the underlying story is not visible) - one needs the extensive form to discover it.
Example.

- Firm E

\[
\begin{bmatrix}
0 \\
2
\end{bmatrix}
\]

- Firm \( \mathcal{I} \)

\[
\begin{bmatrix}
-3 & 2 \\
-1 & 1
\end{bmatrix}
\]

Strategy Sets:

Firm E: \{Out, In\}

Firm \( \mathcal{I} \): \{F if In, A if In\}
Normal form:

\[
\begin{bmatrix}
\text{F if In} & \text{A if In} \\
\text{Out} & 0, 2 & 0, 2 \\
\text{In} & -3, -1 & 2, 1
\end{bmatrix}
\]

Two pure strategy NE:

NE1: (In, A if In)

NE2: (Out, F if In).

NE2 is not credible.

In this Nash equilibrium, what firm I’s strategy says it will do at the unreached node can actually ensure that firm E, taking firm I’s strategy as given, wants to play "out" (even though, given firm E’s strategy, firm I’s strategy choice does not really make a difference to firm I’s pay-off).
* Principle of Sequential Rationality: A player’s strategy should specify optimal actions at every point in the game tree.

At each decision node in the tree, a player should choose an action that is optimal from that point on, given the strategies of other players.

Sequential rationality violated by NE2 in both examples above.

NE1 is sequentially rational in both examples: credible.
Work backwards from the last stage of the game i.e., decision nodes whose only successor nodes are terminal nodes).

Solve for optimal behavior at each such decision node.

Then go to previous stage (i.e., nodes preceding the above) and figure out optimal action of decision maker (while fixing continuation play in the successor nodes to the optimal actions derived previously)

& so on until one reaches the beginning of the game.

This is called backward induction.
Example.

\[
\begin{array}{c}
\bullet 1 \\
L \downarrow \quad \downarrow R \\
3 \bullet \\
\downarrow l \quad \downarrow r \\
\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}
\end{array}
\begin{array}{c}
\bullet 2 \\
\downarrow a \quad \downarrow b \\
\begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix}
\end{array}
\begin{array}{c}
3 \bullet \\
\downarrow l \quad \downarrow r \\
\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}
\end{array}
\begin{array}{c}
3 \bullet \\
\downarrow l \quad \downarrow r \\
\begin{bmatrix} 0 \\ -1 \\ 7 \end{bmatrix}
\end{array}
\begin{array}{c}
-2 \\
-2 \\
o \end{array}
\]
Solving for optimal action in last stage leads to the following reduced form two-stage game:

\[ \begin{array}{c}
\bullet 1 \\
L & \ \ \ \ \ \ \ R \\
\begin{bmatrix}
-1 \\
5 \\
6
\end{bmatrix} & \bullet 2 \\

a & \ \ \ \ \ \ \ b \\
\begin{bmatrix}
5 \\
4 \\
4
\end{bmatrix} & \begin{bmatrix}
0 \\
-1 \\
7
\end{bmatrix}
\end{array} \]
Reduced form first stage game:

\[
\begin{array}{c|c}
L & R \\
\hline
-1 & 5 \\
5 & 4 \\
6 & 4 \\
\end{array}
\]

Solution by backward induction:

\([R, a \text{ if } R, (r \text{ if } L, r \text{ if } R \text{ and } a, l \text{ if } R \text{ and } b)].\]

Can check that this is a NE.

There are two other NE that are not sequentially rational.
Proposition: Every finite game of perfect information has a pure strategy NE that can be derived through backward induction. Moreover, if no player has the same pay-offs at any two distinct terminal nodes, then there is a unique NE that can be derived in this manner.
Games of Complete but possibly Imperfect Information.

How to apply sequential rationality if decision nodes are not necessarily singletons i.e., players do not necessarily observe all predecessor moves.

Example:

Consider the entrant-incumbent game with one modification.

If entrant form decides to play "In" (i.e., enter), entrant and incumbent play a simultaneous move game where they both choose whether to fight or accommodate.
• Firm E

\[
\begin{bmatrix}
0 \\
2
\end{bmatrix}
\]

Firm I • ─ ─ ─ ─ ─ ─ ─ ─ •

\[
\begin{bmatrix}
-3 \\
-1
\end{bmatrix}
\begin{bmatrix}
1 \\
-2
\end{bmatrix}
\begin{bmatrix}
-2 \\
-1
\end{bmatrix}
\begin{bmatrix}
3 \\
1
\end{bmatrix}
\]
Normal form:

Row player: E

Column Player: \( \mathcal{I} \)

\[
\begin{bmatrix}
\text{Out, A if In} & 0, 2 & 0, 2 \\
\text{Out, F if In} & 0, 2 & 0, 2 \\
\text{In, A if In} & 3, 1 & -2, -1 \\
\text{In, F if In} & 1, -2 & -3, -1 \\
\end{bmatrix}
\]
Three pure strategy NE in the normal form game:

NE1: [(Out, A if In), (F if In)]

NE2: [(Out, F if In), (F if In)]

NE3: [(In, A if In), (A if In)]
Extensive form equivalent to:

- Firm E

\[
\begin{array}{c}
\text{Out} \\
\text{In}
\end{array}
\quad \begin{bmatrix}
0 \\
2
\end{bmatrix}
\begin{bmatrix}
A & 3, 1 & -2, -1 \\
F & 1, -2 & -3, -1
\end{bmatrix}
\]

In the simultaneous move (sub)game that follows "In" (captured in the matrix), unique NE: (A,A).

Thus, firm E should expect that if it enters, they will both play (A,A).

So, firm E should choose 'In'.

Only NE3 is a reasonable prediction of the game.
Subgame Perfect Nash Equilibrium. [Selten, 1965].

**Definition.** A subgame of an extensive form game is a subset of the game having the following properties:

(i) It begins with an information set containing a single decision node, contains all the decision nodes that are successors (both immediate and later) of this node, and contains only these nodes;

(ii) If a decision node \( x \) is in the subgame, then every other node contained in the information set containing \( x \) is also in the subgame (no broken information sets).
The entire game is also a subgame.

A subgame is an extensive form game in its own right and one can apply all of the equilibrium/solution concepts - including Nash equilibrium to it.
* A strategy profile $\sigma$ in extensive form game $\Gamma_E$ is said to induce a Nash equilibrium in a particular subgame of $\Gamma_E$ if the moves specified by $\sigma$ for information sets within the subgame constitute a Nash equilibrium when this subgame is considered in isolation.

**Definition.** A profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ in an $I$-player extensive form game $\Gamma_E$ is a subgame perfect Nash equilibrium (SPNE) if it induces a Nash equilibrium in every subgame of $\Gamma_E$. 
By definition, every SPNE is a NE but the converse is not true.

SPNE: the most widely used refinement of NE in economic applications.* If the only subgame of a game is the game as a whole, then every NE is subgame perfect.
* A SPNE induces a SPNE in every subgame of the game.
* In finite games of perfect information, the set of SPNE coincides with the set of NE that can be derived through backward induction procedure.

[Why? Every decision node initiates a subgame.

Single decision maker at each level.

Consider decision nodes at the last stage of game: NE is simply the optimal action.

Backward induction: decision nodes before end of game, best response to optimal actions in the end decision nodes.

So, backward induction leads to NE in the subgames beginning from the decision nodes preceding the end decision nodes.

And so on...]
**Proposition:** Every finite game of (complete and) perfect information has a pure strategy SPNE. Moreover, if no player has the same payoffs at any two terminal nodes, there is a unique SPNE.
Generalized backward induction to solve for SPNE in more general finite dynamic games (not necessarily perfect information):

1. Look at the final subgames at the end of the game tree (no further nested subgame) and solve for NE.

2. Select one NE for each of them and replace the final subgames in the game tree by terminal payoffs equal to the NE payoffs of the players (in the relevant final subgames).

This is called the reduced game.

3. Now, repeat this for the reduced game & continue doing this until the moves at all information sets of the original game have been determined.

The strategies that specify the collection of moves obtained through this process constitute a SPNE.
If multiple Nash equilibria are never encountered in this
generalized backward induction process, then this profile
of strategies is the unique SPNE.

If multiple NE are encountered, the full set of SPNE is
identifying the procedure for each possible equilibrium
that could occur at the subgames.
Example.

- Firm E

\[
\begin{bmatrix}
0 \\
2
\end{bmatrix}
\]

- Firm E

\[
\begin{bmatrix}
-6 \\
-6
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 \\
-1
\end{bmatrix}
\]

\[
\begin{bmatrix}
-3 \\
-3
\end{bmatrix}
\]
- Firm E

\[
\begin{bmatrix}
0 \\
2
\end{bmatrix}
\begin{bmatrix}
S & L \\
S & -6, -6 & -1, 1 \\
L & 1, -1 & -3, -3
\end{bmatrix}
\]

Two NE in the last subgame: \((S, L), (L, S)\).
Two reduced games:

1)  
   - Firm E

   \[
   \begin{bmatrix}
   0 & 1 \\
   2 & -1
   \end{bmatrix}
   \]

2)  
   - Firm E

   \[
   \begin{bmatrix}
   0 & -1 \\
   2 & 1
   \end{bmatrix}
   \]

Two SPNE:

\{ (In, L if In), S if In \} 

\{ (Out, S if In), L if In \}.
**Finitely repeated simultaneous move game.**

Consider a normal form game (simultaneous move game) \( \Gamma_N \) which is played repeatedly for a finite \( T \) number of times.

The normal form game which is played repeatedly is called a "(one) stage game" (or "one shot game").

Allow players to play mixed "strategies" in the one stage game if they wish.

After each round, all players observe the (pure) strategies actually played in the previous round and then play the next round.

This entire \( T \)-stages game is a dynamic game of complete and imperfect information.
For $t > 1$, let $h_t$ be the history of the game (i.e., what players played) as observed till the end of $(t - 1)$ rounds of play and before the $t^{th}$ round is played.

Each possible $t = 1, \ldots, T$, and each possible $h_t$ (for $t > 1$) defines a distinct decision node for each player.

In this dynamic game, the strategy of a player specifies what she is going to choose for each $t = 1, \ldots, T$, and each possible $h_t$ (for $t > 1$).

Payoff in the dynamic game: sum of payoffs over the $T$-stages (can also look at discounted sum).
Proposition. If the stage game $\Gamma_N$ has a unique NE, then there is a unique SPNE of the game where for all $t = 1, ... T$, and all $h_t$, players play the NE of the stage game.
More generally:

**Proposition.** If the stage game $\Gamma_N$ has multiple NE, then any strategy profile of the dynamic game where for each $t = 1, \ldots, T$, players play one of the NE of the stage game independent of $h_t$, is a SPNE.
However, if $\Gamma_N$ has multiple NE, then there may be SPNE where players do not play any of the NE of the stage game $\Gamma_N$ for some $t$. 
**Example.**

Suppose the following normal form game is repeated twice.

Payoff : sum of payoffs in the two stages.

\[
\begin{bmatrix}
T & L & C & R \\
1 & 1 & 5.0 & 0.0 \\
0 & 1 & 4.4 & 0.0 \\
0 & 0 & 0 & 3.3 \\
\end{bmatrix}
\]

The stage game has two NE: \((T, L), (B, R)\).

SPNE1: \{((T in stage 1, and in stage 2, play T whatever be the history), (L in stage 1, and in stage 2, play L whatever be history))\}
SPNE2: \{ (B \text{ in stage 1, and in stage 2, play } B \text{ whatever be the history}), (R \text{ in stage 1, and in stage 2, play } R \text{ whatever be history}) \} \\

SPNE3: \{ (T \text{ in stage 1, and in stage 2, play } B \text{ whatever be the history}), (L \text{ in stage 1, and in stage 2, play } R \text{ whatever be history}) \} \\

SPNE4: \{ (B \text{ in stage 1, and in stage 2, play } T \text{ whatever be the history}), (R \text{ in stage 1, and in stage 2, play } L \text{ whatever be history}) \} \\

These four SPNE correspond to playing some NE of the stage game in each period.
SPNE5:

\[
\begin{bmatrix}
L & C & R \\
T & 1,1 & 5,0 & 0,0 \\
M & 0,5 & 4,4 & 0,0 \\
B & 0,0 & 0,0 & 3,3 \\
\end{bmatrix}
\]

Strategies:

Player 1: Play \( M \) in stage 1.

In stage 2, play \( B \) if \((M, C)\) has been played in stage 1 and play \( T \), otherwise.

Player 2: Play \( C \) in stage 1.

In stage 2, play \( R \) if \((M, C)\) has been played in stage 1 and play \( L \), otherwise.
Generalized backward induction.

Subgames in the second stage are of two types:

(i) The one following \((M, C)\) being played in stage 1

(ii) The ones following \((M, C)\) not being played in stage 1

The indicated strategies induce NE in both classes of subgames.
The reduced game in stage 1 (given the above strategies):

\[
\begin{bmatrix}
T & L & C & R \\
2,2 & 6.1 & 1,1 \\
M & 1,6 & 7,7 & 1,1 \\
B & 1,1 & 1,1 & 4,4
\end{bmatrix}
\]

The specified strategies for the first stage clearly a NE in the reduced game.

Thus, this is a SPNE.
In SPNE5, players do better than the if they played the best (or Pareto efficient) NE of the stage game twice.

They behave cooperatively in the first round (even though playing cooperatively i.e., \((M, C)\) is not a NE in the one stage game).

In stage 2 (last period), players must play one of the two NE of the stage game as it is essentially a one shot game.

However, multiplicity of NE here allows players to incorporate the threat of playing the bad NE rather than the good one in case they do not play cooperatively in the first round.

This is a credible threat (if we ignore renegotiation possibilities).

This illustrates: *finitely repeated interaction can induce "cooperation" in early periods when there are multiple NE in the stage game.*
In infinitely repeated games, we will see that there are SPNE that involve cooperative play even though there is a unique NE in the stage game.
A deep problem with sequential rationality.

SPNE: players should play an SPNE wherever they find themselves in the game tree, even after a sequence of events that is contrary to the prediction of the theory (i.e., the actual play that ought to be induced if the players played the SPNE strategies).
Example. (Centipede game).

Finite game of perfect information.

2 players 1 & 2.

Each player starts with $1 in front.

They alternate saying "stop" or "continue".

When a player (whose turn it is to move) says "continue", $1 is taken by a referee from her pile and $2 is added to rival’s pile.

When a player says "stop", play is terminated and each player receives the money currently in her pile.

Play stops in any case if both players' pile reaches $100.
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<td>2</td>
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\[
\begin{bmatrix}
0 \\
3
\end{bmatrix}
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\[
\begin{bmatrix}
2 \\
2
\end{bmatrix}
\]

.............
Backward induction: unique SPNE is that players choose stop at each decision node where they are asked to move.
In actual play of this SPNE, play will end in the first move with player 1 stopping the game and both players getting $1 each.

Really bad outcome, considering that they could get $100 each if they played to continue every time.
Is the SPNE a reasonable prediction?

Player 1 says stop in stage 1, because she thinks player 2 will choose stop at her first turn.

But if player 1 *thinks that* either:

(i) player 2 is not fully rational and therefore does not compute the SPNE by backward induction

or (ii) player 2 is rational but does not know whether player 1 is rational and thus, observing player 1 choose to continue (when SPNE says player 1 should play stop), she supposes that player 1 is not rational (in the sense of playing SPNE) and therefore, if she chose to continue, player 1 would not stop the game in the next stage but would actually continue it further allowing player 2 to move again

- then it may be optimal for an actually rational player 1 to continue in the first stage.

Note that arguments like (i) or (ii) involve some contradiction to common knowledge of rationality.
SPNE denies this possibility - however, it then leaves open the question of how players think about the game and other players when they find themselves at decision nodes that ought not be reached if players played by sequential rationality.

One resolution: treat deviations from SPNE as mistakes that occur with extremely small probability and unlikely to be repeated again.

This is the approach taken by a somewhat different refinement concept called "trembling hand perfection".