Time Value of Money

- **Future Value**: The process of going from today’s values, or present values (PVs), to future values (FVs) is called compounding. To illustrate, suppose you deposit $100 in a bank that pays 5% interest each year.

  - The future value at the end of 1 year is given by
    \[ FV_1 = 100(1 + 0.05) = 105 \]
  - The future value at the end of 2 years is given by
    \[ FV_2 = 105(1 + 0.05) = 110.25 \]

  Proceeding similarly, we have

  \[
  
  FV_3 = 110.25(1 + 0.05) = 115.76 \\
  FV_4 = 115.76(1 + 0.05) = 121.55 \\
  FV_5 = 121.55(1 + 0.05) = 127.63 
  \]

  In general, the future value of a $100 invested today at a rate of return \( i\% \) is given by

  \[ FV_n = 100(1 + i\%)^n \]

  We will call the number \( (1 + i\%)^n \) as \( (FVIF)_{i\%,n} \) (see Table 3) and write

  \[ FV_n = 100 \ (FVIF)_{i\%,n} \]

- **Present Value**: The present value of a cash flow due \( n \) years in the future is the amount which, if it were on hand today, would grow to equal the future amount. Since $100 would grow into $127.63 in 5 years at a 5% interest rate, $100 is the present value of $127.63 due in 5 years when the opportunity cost rate is 5%.
• Note that we have the following relationship for Future and Present values:
\[ FV_n = PV(1 + i\%)^n \]
which implies
\[ PV = FV_n \left( \frac{1}{1 + i\%} \right)^n \rightarrow PV = FV_n (PVIF)_{i\%,n} \]
i.e., we call the number \( \left( \frac{1}{1 + i\%} \right)^n \) as \((PVIF)_{i\%,n}\) (Present Value Interest Factor for \(i\) and \(n\)) (see Table 1).

• Let us now confirm that the $100 is the present value of $127.63 due in 5 years when the opportunity cost rate is 5%. We have (from Table 1)
\[ PV = $127.63 (PVIF)_{5\%,5} = $127.63(0.784) = $100. \]

**Example 1:** Suppose you are to receive the following cash flow stream and the one period discount rate (opportunity cost rate) is 10%. What is the present value of this cash flow stream?

<table>
<thead>
<tr>
<th>i=10%</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$2500</td>
<td>$4000</td>
<td>$5000</td>
<td></td>
</tr>
<tr>
<td>PV=?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ PV = 2500 (PVIF)_{10\%,2} + 4000 (PVIF)_{10\%,3} + 5000 (PVIF)_{10\%,4} \]
\[ = 2500(0.826) + 4000(0.751) + 5000(0.683) \]
\[ = $8484 \]
• **Present Value of an Annuity:** An annuity is a series of equal payments made at the end of fixed intervals. For example, $100 at the end of each of the next three years is a three year annuity. Payments on mortgages, car loans and students loans are typically set up as annuities.

• **Example 2:** Below is a 4 year annuity of $100, discounted at 5%.

<table>
<thead>
<tr>
<th>i=5%</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>PV=?</td>
<td>$100</td>
<td>$100</td>
<td>$100</td>
<td>$100</td>
</tr>
</tbody>
</table>

We can find the present value of the above annuity by individually discounting each payment of $100. But there is an easier way. The present value is given by (See Table 2)

\[
PV = 100 \ (PVIFA)_{5%,4}
= 100 \times 3.546
= 354.6
\]

• **Example 3:** Below is a 15 year annuity of $1000, discounted at 8%.

<table>
<thead>
<tr>
<th>i=8%</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>PV=?</td>
<td>$1000</td>
<td>$1000</td>
<td>$1000</td>
<td>$1000</td>
</tr>
</tbody>
</table>

The present value is given by (See Table 2)

\[
PV = 1000 \ (PVIFA)_{8%,15}
= 1000 \times 8.559
= 8559
\]

**Example 4 (Missing Cash Flows):** Suppose that you have an investment opportunity that pays $300 at t=1, $300 at t=2, $300 at t=3 and some fixed cash flow at the end of each of the remaining 17 years (see below).
The present value of this cash flow stream is \( PV = \$5500 \), and an alternative investment of equal risk has a required rate of return of 9%. What is the annual cash flow received at the end of each of the final 17 years?

**Answer to Example 4:** The present value of the above cash flow stream can be written as

\[
PV = 300(PVIFA)_{9\%,3} + x(PVIFA)_{9\%,17}(PVIF)_{9\%,3}
\]

\[
\Rightarrow 5500 = 300(2.531) + x(8.544)(0.772)
\]

\[
\Rightarrow 5500 = 759.3 + 6.59x
\]

\[
\Rightarrow x = \$718.7
\]

**Example 5: (Amortized Loan):** Suppose that you borrowed \$60,000 from a bank. The loan should be paid back in 4 years and the bank proposed the following payment plan:

**Year 1:** \$20,000, **Year 2:** \$20,000, **Year 3:** \$20,000, **Year 4:** \$20,000

What is the interest rate on the loan?

\[
\$60,000 = 20,000(PVIFA)_{i\%,4}
\]

\[
\Rightarrow \frac{\$60,000}{20,000} = (PVIFA)_{i\%,4}
\]

\[
\Rightarrow 3 = (PVIFA)_{i\%,4}
\]

From Table 2, \( i = 12\% \)