1 Optimal Portfolio with One Risky and One Safe Asset

Consider an investor with an initial wealth $w_0$. The investor has mean-variance preferences over his final wealth $w$ and wants to maximize

$$U = E(w) - \frac{AVar(w)}{2}$$

There are two assets. A risky asset has a share price $p$. Each share yields an uncertain return $X$ which is distributed as

$$X \sim N(\mu, \sigma_p^2)$$

Each dollar invested in the safe asset yields a certain gross return $r$. Accordingly, if the investor buys $d$ shares of the risky asset and invests the rest of his wealth in the safe asset, the budget constraint takes the form

$$w_0 = pd + F$$

where $F$ is the dollar amount invested in the safe asset. Then, the investor’s final wealth distribution becomes

$$w = dx + rF$$

which can be written as

$$w = dx + r(w_0 - pd)$$

$$\Rightarrow w = rw_0 + d(x - rp)$$

Therefore we have

$$E(w) = rw_0 + d(\mu - rp)$$

and

$$Var(w) = d^2 \sigma_p^2$$

The investor chooses the risky asset position $d$ to maximize

$$U = E(w) - \frac{AVar(w)}{2} = rw_0 + d(\mu - rp) - \frac{Ad^2 \sigma_p^2}{2}$$

which yields

$$d^* = \frac{\mu - rp}{A\sigma_p}$$