Hedging with an Index Option

- **Problem**: You are a portfolio manager with a stock portfolio of size $P$ and beta equal to $\beta$. You want to make sure that the portfolio will not lose more than $z\%$ in value. In other words, you want to insure $(100 - z)\%$ of your portfolio value.

- **Example**: $P = $10 million, $\beta = 1.5$ and you want to insure 90\% of the portfolio value, i.e., $z = 10\%$.

- **Hedge Instrument**: We will use a put option on a stock index (like S&P 500). This is a put option where the underlying asset is the index. It gives the long position the right to sell the index at a strike price $\$X$. If the stock index at expiration is $S_T$, and if $X > S_T$, the put will be exercised and the profit to the long put position will be

  $$
  \text{profit} = \$100(X - S_T)
  $$

  where 100 is just an institutional convention. Of course, if $X < S_T$, put is not exercised and the payoff is zero.

- **If you are trying to hedge $(100 - z)\%$ of your portfolio with size $P$ and beta equal to $\beta$, how many put options on the stock index should you go long? and what is the strike price $X$ of the put you should use?**

- **Deriving the optimal hedge**

  - Suppose the current index today is $S_0$. The strike price $X$ of the put option on the index must be

    $$
    X = S_0(1 - \frac{z\%}{\beta})
    $$

  - Now let’s derive the number of put options to go long. The current index is $S_0$. Suppose the index falls by $\Delta\%$ so that $S_T$, the index value at expiration

    $$
    S_T = (1 - \Delta\%)S_0
    $$
If the index falls by $\Delta\%$, your portfolio value will fall by $\beta\Delta\%$. So the value loss in your portfolio is

$$\text{Loss in Portfolio Value} = P\beta\Delta\%$$

Since you want to hedge $(100 - z)\%$ of your portfolio, the excess loss you need to cover (by earnings from the put option) is given by

$$\text{Excess Loss to Cover} = P\beta\Delta\% - Pz\%$$ \hspace{1cm} (4)

Now, let’s compute the profit from one put option on the index. Remember that

$$S_T = (1 - \Delta\%)S_0$$

$$X = S_0(1 - \frac{z\%}{\beta})$$

Therefore, the profit from one put option on the index is

$$\text{profit} = 100(X - S_T)$$

$$= 100[S_0(1 - \frac{z\%}{\beta}) - (1 - \Delta\%)S_0]$$

$$= \frac{100S_0(\beta\Delta\% - z\%)}{\beta}$$

Accordingly, the number of puts to cover the excess loss $P\beta\Delta\% - Pz\%$ is

$$\# \text{ of Puts} = \frac{P[\beta\Delta\% - z\%]}{[100S_0(\beta\Delta\% - z\%)]/\beta} = \frac{\beta P}{100S_0}$$

**Conclusion:** Suppose the current index is $S_0$. If a portfolio manager with a stock portfolio of size $P$ and beta equal to $\beta$ wants to make sure that the portfolio will not lose more than $z\%$ in value (in other words, he/she wants to insure $(100 - z)\%$ of the portfolio value), then he/she should go long in $\frac{\beta P}{100S_0}$ put options on the index with a strike price of $X = S_0(1 - \frac{z\%}{\beta})$. 

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Review Problems on Hedging Stock Portfolio with a Put Option on S&P-500 Index

- **Problem 1**: Suppose you have a stock portfolio worth $200,000. The portfolio beta is 2.5. The current S&P-500 Index is 1000.

  - a) If the index goes down by 10%, what is the expected dollar loss on your portfolio?
    
    * For a beta of 2.5, if the market index goes down by 10%, portfolio value is expected to go down by
      
      \[ \beta(10\%) = 2.5(10\%) = 25\% \]

      Therefore the expected dollar loss on the portfolio is
      
      \[ (200,000)(0.25) = 50,000 \]

  - b) Suppose your tolerable loss on the portfolio is 10%. In other words, you want to hedge 90% of your portfolio value. What is your expected excess loss in dollar terms if index goes down by 10%?

    
    * Maximum tolerable loss = \((200,000)(0.10) = 20,000\)

    Therefore, the excess loss (if the index goes down by 10%) is

    \[ \text{Excess Loss} = 50,000 - 20,000 = 30,000 \]

  - c) Suppose your tolerable loss on the portfolio is 10%. How many put options on the S&P-500 Index should you buy? What should be the strike price of the put options?

    * The strike price of the put options must be
      
      \[ X = S_0\left(1 - \frac{z\%}{\beta}\right) = 1000\left(1 - \frac{10\%}{2.5}\right) = 960 \]

      which implies that the expected profit from one put option (if the index goes down by 10%) is

      \[ $100(X - S_T) = $100(960 - 900) = 6000 \]
Therefore, the number of put options to cover the excess loss of $30,000 is

\[ \text{# of puts} = \frac{$30,000}{5} = 5 \]

**Problem 2**: Suppose you have a stock portfolio worth $500,000. The portfolio beta is 2. The current S&P-500 Index is 1000. You want to hedge 80% of the portfolio. In other words, your tolerable loss is \( z = 20\% \).

- a) If the index goes down by 20 \%, what is the expected dollar loss on your portfolio?
- b) Given that your tolerable loss is \( z = 20\% \), what is the excess loss that you have to cover if the index goes down by 20 \%?
- c) Suppose your financial advisor suggested that you should buy put options on the S&P-500 Index with strike price \( X=900 \). If the index goes down by 20\%, what is your payoff from one such put option?
- d) How many put options should you buy with strike \( X=900 \) to cover the excess loss you found in part b?