An Introduction to Radiation Boundary Conditions for Time-Domain Scattering Problems

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Why are waves so important in science and engineering?
Their fundamental feature is the ability to propagate long distances relative to the wavelength and thus provide detailed information about the objects they've encountered or the events which produced them.

What's hard about wave propagation problems?

Typical applied problems are posed in
COMPLEX, UNBOUNDED domains.

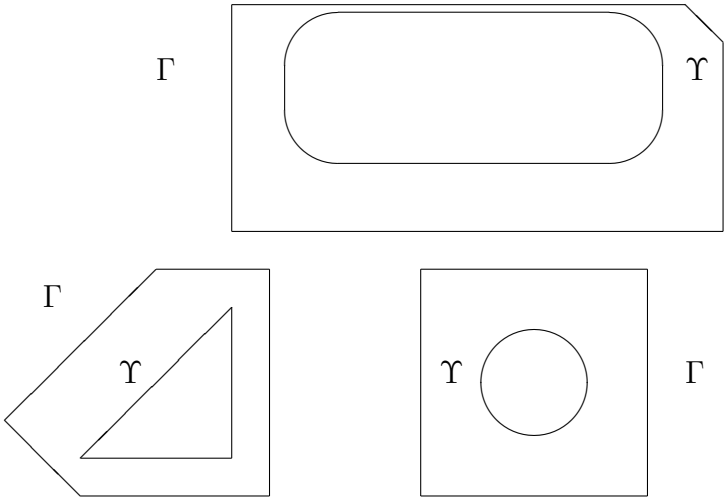
There is little smoothing of the solution in space-time.

An "ideal" numerical wave simulator should include:

Radiation boundary conditions which provide arbitrary accuracy at small cost (spectral convergence, weak dependence on the simulation time and wavelength)

Algorithms for using the information at or near the boundary to directly propagate the solution to remote locations.

Reliable high-resolution volume discretizations applicable in complex geometry (i.e. on grids that can be generated efficiently).



Domains for a multiple scattering problem: Υ is the computational domain, Γ is the artificial boundary.

A general approach to these problems is the **construction of approximate solution representations** in the far field. Using these representations we can

Apply a radiation boundary operator, $\mathcal{B}$, on the artificial boundaries. That is find some $\mathcal{B}$ which leads to a well-posed problem and annihilates functions in some truncation/approximation to the representation.

Extend the solution to the far field (receivers, other scatterers) by evaluating a truncation/approximation to the representation.

The focus of this talk will be methods based on plane waves and variants, effectively computed via Fourier-Laplace transforms. There are, however, alternatives, some of which I will mention below. For a longer (but by no means exhaustive) list of references see the review articles (Acta Numerica 1999, J. Comput. Math. 2007).

Equivalent source representations

$$u = \int \frac{g(t - |x - \rho|, \rho)}{\rho} d\rho \approx \sum_k \frac{G_k(ct - r_k)}{r_k}$$

Difference potential method of Ryaben'kii and Tsynkov (JCP 2001) - construct an auxiliary function satisfying a forced wave equation in free space which agrees with the solution of the original problem at the artificial boundary.

$$\phi = \eta u, \quad \eta = \begin{cases} 0 & x \in \Upsilon_1 \\ 1 & x \in \mathbb{R}^d - \Upsilon_2 \end{cases}$$

where $\Upsilon_1 \subset \Upsilon_2 \subset \Upsilon$ Then

$$L\phi = g(u), \quad g = 0, \quad x \ni \Upsilon_2 - \Upsilon_1$$

Fast algorithms with a different method for constructing the sources have been developed by Bruno and Hoch. (See Hoch's dissertation at Caltech, 2008.) Here the source locations are restricted to a sparse lattice and FFTs can be employed. The G_k are determined by a least squares fit to time-gated data in the interior.

Multipole expansions

$$u = \sum_k \frac{g_k(ct - r, \theta\phi)}{r^k}$$

Theoretical development by Heyman and Devaney (J. Math. Phys. 1996). First proposed for the construction of local radiation boundary condition sequences by Bayliss and Turkel (Commun. Pure and Appl. Math. 1980). Localized construction for arbitrary order boundary conditions using auxiliary functions on the sphere derived by H. and Hariharan (Appl. Num. Math. 1998), field extension algorithms and multiple scattering applications recently developed by Sim and Grote. (See Sim's Ph.D. dissertation at the University of Basel, 2009.) The idea in these algorithms is to write down three-term recursions relating the tails of these series.

Plane Waves

$$u = \int_S A(\phi, ct - \mathbf{x} \cdot e_\phi)$$

Theoretical development by Heyman (J. Math. Phys. 1996). Basis for the plane wave fast time domain algorithm (PWFTD) of Michielssen and coworkers (JCP 98). Compute the expansion for space-time localized sources using the slant stack transform. Careful time-gating is required to circumvent causality issues - **plane wave representations are not causal - require the inclusion of evanescent waves to represent the complete field.**

Despite this defect plane wave expansions have been the basis used for most constructions of radiation boundary condition sequences (e.g. by Engquist and Majda Math. Comp. 1977, Commun. Pure and Appl. Math. 1979 and by Higdon Math. Comp. 1986, 1987; JCP 1992, SINUM 1994) and absorbing layers (e.g. the perfectly matched layer (PML) by Bérenger JCP 1994, Chew and Weedon Micro. Optic. Technol. Lett. 1994). However, as a consequence of their lack of accuracy for evanescent waves, these all suffer from nonuniform accuracy for large times. (We will make this precise later on.)

When comparing different methods, we will try to estimate the work required (per grid point $\times$ time step) in terms of the error tolerance, ϵ, the scales in the problem such as the wavelength, λ, the separation of the boundary from sources, δ, or scaled simulation time, cT, relative to the size of the scatterer. **Generally we'd like these dependencies to be logarithmic.** This will be my criterion for optimality.

Boundary conditions in a half-space for hyperbolic systems. We consider

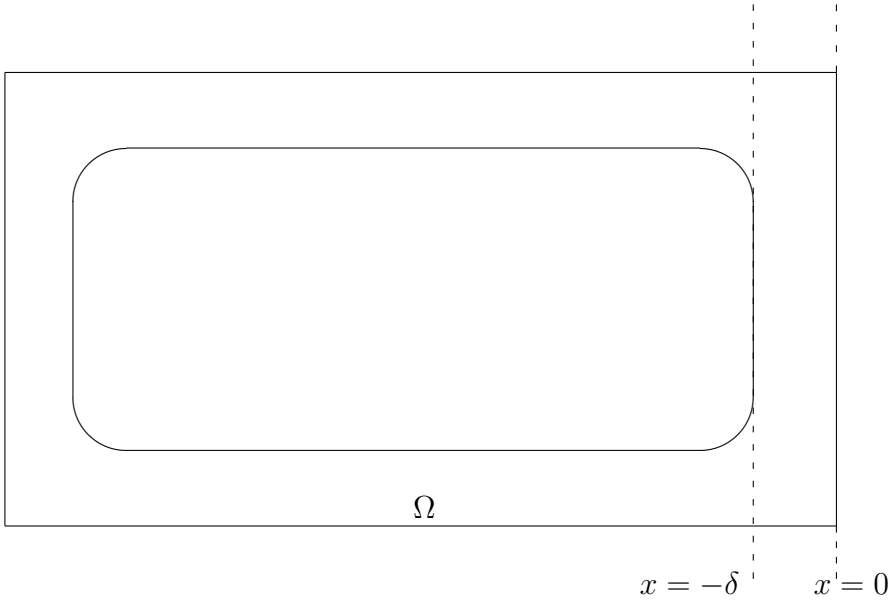
$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + \sum_j B_j \frac{\partial u}{\partial x_j} = 0$$

where $B_j = B_j^T$ and

$$A = \begin{pmatrix} A_- & 0 & 0 \\ 0 & A_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $A_\pm$ are, respectively, positive-definite and negative-definite symmetric matrices. The corresponding partition of the solution, $u = (u_-, u_+, u_0)^T$, corresponds to incoming, outgoing, and tangential variables.

We have block-diagonalized the matrix corresponding to x-derivatives in preparation for deriving boundary conditions on the hyperplane $x = 0$. For our error estimates we will assume that the scatterer is located a (small) distance δ from the boundary.



Computational domain, Ω, for a scattering problem with a rectangular artificial boundary.

Example - acoustics in two space dimensions (scaled so the sound speed is one)

$$\begin{aligned}
\frac{\partial p}{\partial t} + U\frac{\partial p}{\partial x} + V\frac{\partial p}{\partial y} + u_x + v_y &= 0 \\
\frac{\partial u}{\partial t} + U\frac{\partial p}{\partial x} + V\frac{\partial p}{\partial y} + p_x &= 0 \\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} + p_y &= 0
\end{aligned}$$

We put this in normal characteristic form by introducing the variables $r_\pm = p \pm u$. The the matrix, A, takes the form (not necessarily ordered as above)

$$A = \begin{pmatrix} U-1 & 0 & 0 \\ 0 & U+1 & 0 \\ 0 & 0 & U \end{pmatrix}$$

There are many cases to consider based on the size and sign of U. (I'm skipping two of them - $U = 1$, sonic boundary!)

Supersonic inflow $U < -1$: All waves are incoming.

Subsonic inflow $-1 < U < 0$: $p - u$, v incoming, $p + u$ outgoing.

Supersonic outflow $U > 1$: All waves are outgoing.

Subsonic inflow $0 < U < 1$: $p - u$, incoming, $p + u$, v outgoing.

No normal flow $U = 0$: $p-u$, incoming, $p+u$ outgoing, v tangential

Finally its time to write down the answer. We do it by Fourier-Laplace (in $y-t$) transformations. Here s is dual to t and k is dual to y (a vector in higher space dimensions). To properly label waves as causally incoming and outgoing (produced by sources to the left or right of the boundary), it is important to take $\Re s > 0$. To develop approximations we will take

$$\Re s = \frac{1}{T}$$

where T is a time scale over which we want to guarantee accuracy - for example the simulation time. Alternatively one can take $\Re s = 0$ but then use group velocity rather than phase velocity to label solutions as incoming or outgoing.

Solutions:

$$\hat{u} = e^{\mu(s,k)x}\phi$$

$$\left(sI + \mu A + \sum ik_j B_j\right)\phi = 0$$

We label waves by the sign of $\Re\mu$. A simple argument shows the number of solutions of each type is independent of s and k so long as $\Re s > 0$. The idea is that for $\phi^*\phi = 1$

$$\Re s + \Re\mu \cdot \phi^* A\phi = 0$$

Therefore $\Re\mu \neq 0$, and the number of solutions with positive (negative) real part equals the dimension, n_-, (n_+) of A_- (A_+).

We can now write down the **exact** radiation boundary condition at $x = 0$. In words, we can't admit incoming waves. In math, let $B_-(s, k)$ denote any $n_- \times n$ matrix whose null space is the span of the eigenvectors (and generalized eigenvectors) corresponding to the eigenvalues with positive real part. Then:

$$B_- \hat{u} = 0$$

For the supersonic cases this is easy. At supersonic outflow $n_- = 0$ so no boundary condition is needed. At supersonic inflow we can take $B_- = I$ so that

$$u = 0$$

is exact. In the subsonic case, however, B_- will be the symbol of an integrodifferential operator in space and time. That's where we will have work to do!

For simplicity I will take the no-flow case, $U = V = 0$. (For generalizations to problems with flow see H., Bécache, Givoli and Stein CICP to appear.)
We can always take B_- in the form

$$B_- = \begin{pmatrix} I & B_-^+ & B_-^0 \end{pmatrix}$$

and here the eigenvalue problem reduces to solving a quadratic equation. After some elementary algebra we get (nonunique)

$$(s+\gamma)\hat{r}_- + (s-\gamma)\hat{r}_+ = 0$$

$$\gamma = \left(s^2 + k^2\right)^{1/2}$$

We can explicitly invert the transforms to get a representation of the exact condition in integral form:

$$\frac{\partial r_-}{\partial t} + \mathcal{F}^{-1}\left(k^2(J_1(kt)/t) * (\mathcal{F}u)\right) = 0$$

This may look hopelessly expensive due to the space-time integrals, but in fact we (and others) have constructed a fast, low-memory compression of the time integral operator (Alpert, Greengard, H. (SINUM 2000, JCP 2002), Lubich and Schädle, (SISC 2002), Hiptmair and Schädle, (Computing 2003)). Thus the complexity of applying the exact condition is:

$$O\left(\ln\frac{1}{\epsilon}\cdot\ln\frac{cT}{\lambda}\right)$$

The problem is that the construction of the nonlocal conditions breaks down if there are corners - restricted to waveguides and spheres. (For spheres we have the added difficulty of computing spherical harmonic transforms.)

Geometrically more flexible approaches are based either on perfectly matched layers or auxiliary-variable formulations of radiation boundary condition sequences (e.g. H. and Warburton, Wave Motion,2004). Although the typical constructions of these methods look very different, we will see that, in fact, they are closely related. **Optimal local radiation boundary condition sequences correspond to semidiscretizations of PMLs on optimal grids**.

This idea was first put forth by Asvadurov, Druskin, Guddati and Knizherman (SINUM 2003) - however they optimized only considering propagating plane waves. In our approach we will construct conditions to minimize the error. **Our main result is to show how local radiation boundary conditions can be constructed which are optimal in that their complexity will scale be**

$$O\left(\ln\frac{1}{\epsilon}\cdot\ln\frac{cT}{\delta}\right)$$

We can realize these rational interpolants using auxiliary variable formulations of local radiation boundary conditions or, equivalently, semidiscretized PMLs. As PMLs are somewhat standard, let's look at briefly their construction following Appelö, H. and Kreiss (SIAP 2006). The idea is simply to write down equations whose **solutions** are given by

$$\hat{u} = e^{r(s,k)\cdot\mu(s,k)x}\phi$$

$$\left(sI + \mu A + \sum ik_j B_j\right)\phi = 0$$

where $r(s,k)$ is a rational function chosen to increase

$$|\Re(r\mu)|$$

and the **equations** are modifed so that the modal eigenfunctions ϕ are identical to those in the original domain. The latter condition ensures a reflecionless interface (perfect matching). The equation modification then is

$$\frac{\partial}{\partial x} \rightarrow \frac{1}{r(s,k)}\frac{\partial}{\partial x}$$

implemented in practice by adding auxiliary variables.

A standard choice for r for an isotropic system is

$$\frac{1}{r} = 1 + \eta + \frac{\sigma}{s + \sigma + \alpha}$$

where σ is the absorption parameter, η is the stretching parameter and α is the complex frequency shift parameter. Typically σ and μ are made functions of x.

This standard layer is primarily built to absorb propagating modes, so it also suffers from long-time accuracy loss. With $\alpha = 0$ Diaz and Joly (CMAME 2006) have also been able to compute the exact error for planar PML leading to the estimate

$$(1 + \eta)^2 L^2 \sigma \propto \ln \frac{1}{\epsilon} \cdot \left(c^2 T\right)$$

where L is the layer width. (The estimate holds for variable coefficient layers using average values.) Thus assuming a well-resolved layer PML is also suboptimal for long time with a complexity growing like $\sqrt{T}$ - to get optimal performance one must use stretched grids.

Given this fact it makes sense to directly work with a semidiscretized layer and not worry at all about consistent approximation to the PML equations. Instead, we will simply try to force our layer (or equivalently our radiation boundary condition) to be exact at the optimal interpolation nodes mentioned earlier. Thus we'll call what we use Complete Radiation Boundary Conditions (or CRBCs for short) and construct them by interpolating μ along our inversion contour. We first derive a convenient representation for μ via some elementary computations.

$$\mu = \cos\phi \cdot \bar{s} + \frac{1}{cT} \cdot \frac{\sin^2\phi}{\cos\phi}$$

with $|\phi| < \frac{\pi}{2}$ everywhere on the inversion contour!

As an aside I comment that this formula leads to a new exact representation of solutions ot the wave equation in a half space

$$u = \int_{\mathcal{S}} \int_0^{\frac{\pi}{2}} C(ct - x\cos\phi, s \cdot y, \phi, s) e^{-\frac{x+\delta}{cT} \cdot \frac{\sin^2\phi}{\cos\phi}} d\phi ds.$$

We see that individual terms in the formula are waves with propagation angle ϕ and a decay rate given by the ϕ-dependent exponent.

The idea behind the CRBCs is to interpolate the exact condition at some collection of eigenvalues. We choose the interpolants to be affine in s (and also possibly depending on k) - in general

$$\begin{aligned}\mu_j &= c_j s + \sigma_j - \sum_d i k_d \beta_{jd} \\ \bar{\mu}_j &= -\bar{c}_j s - \bar{\sigma}_j - \sum_d i k_d \bar{\beta}_{jd}\end{aligned}$$

Here we suppose μ_j interpolates an eigenvalue with positive real part and $\bar{\mu}_j$ interpolates an eigenvalue with negative real part.

For acoustics/Maxwell

$$\mu_j = \cos\phi_j \cdot \bar{s} + \frac{1}{cT} \cdot \frac{\sin^2 \phi_j}{\cos\phi_j}$$

$$\bar{\mu}_j = \cos\bar{\phi}_j \cdot \bar{s} + \frac{1}{cT} \cdot \frac{\sin^2 \bar{\phi}_j}{\cos\bar{\phi}_j}$$

We can realize these interpolants by evolving auxiliary variables along the boundary. The idea is to write down recursions which terminate for outgoing waves with $\mu = \mu_j$ or incoming waves with $\mu = \bar{\mu}_j$.

For $j = 0, \ldots, q$ recursively define:

$$\frac{\partial \psi_j}{\partial t} - \left(c_j \frac{\partial}{\partial t} + \sigma_j \right) A\psi_j + \sum_d (B_d + \beta_{jd} A) \frac{\partial \psi_j}{\partial y_d} =$$
$$\frac{\partial \psi_{j+1}}{\partial t} + \left(\bar{c}_j \frac{\partial}{\partial t} + \bar{\sigma}_j \right) A\psi_j + \sum_d (B_d + \bar{\beta}_{jd} A) \frac{\partial \psi_j}{\partial y_d},$$

with $\psi_0 = u$. Truncate by:

$$\psi_{-,q+1} = 0,$$

with $\frac{\partial \psi_{+,0}}{\partial t}$ computed from the interior.

To see that this is actually an interpolant and to derive error estimates we compute the complex reflection coefficient. We take account of the separation, δ, by measuring the data and the reflection at $x = -\delta$.

As mentioned earlier one can reinterpret the recursions as a semidiscretized perfectly matched layer (PML) by reinterpreting the recursion indices as grid indices. If we set $u_j = (-1)^j \psi_j$, $2u_{j+1/2} \sim u_{j+1} + u_j$

$$\frac{\partial u_{j+1/2}}{\partial x} \sim -\frac{\bar{\mu}_j}{2} u_{j+1} - \frac{\mu_j}{2} u_j$$

Recalling the signs of the real parts of these approximate eigenvalues we recognize the right hand side as a frequency-dependent difference formula. Thus an alternate view of CRBCs is that they are an optimal semidiscretization of a (nonstandard) PML.

What we've left out is the choice of nodes to guarantee rapid convergence. We prove (Hagstrom and Warburton SINUM 2009) that interpolation nodes can be chosen so that to guarantee an error less than ϵ up to time T

$$q \propto \ln \frac{1}{\epsilon} \cdot \ln \frac{cT}{\delta}.$$

nodes are needed.

In contrast with the logarithmic dependence on $\frac{cT}{\delta}$ we have with the CRBCs, Diaz and Joly (SIAP 2005) have shown using a direct construction of the reflection via the Cagniard-de Hoop method that for radiation boundary conditions based on propagating plane waves alone (i.e. with the approximation $\mu_j \approx \cos \phi_j \cdot \bar{s}$) one needs

$$q \propto \ln \frac{1}{\epsilon} \cdot \left(\frac{cT}{\delta} \right)$$

We will present a numerical experiment illustrating this poor long time behavior later on.

The estimate is proven using estimates of the complex reflection coefficient adapting Newman's famous construction of exponentially convergent rational approximations to $|x|$ (Mich. Math. J. 1964) combined with Parseval's relation. It is in using Parseval's relation for Laplace transforms that the restriction to $\Re s = O(T^{-1})$ arises.

Directly we show that the claimed estimate holds for approximations of the form:

$$\cos\phi_j = \left(2\frac{\delta}{cT\ln\frac{1}{\epsilon}}\right)^{j/q}, \quad j = 0, \ldots, q.$$

In practice we can use the Remez algorithm to compute optimal nodes. We emphasize that the reflection coefficient in this case directly leads to an error estimate, in contrast with the reflection coefficient for propagating plane waves which is often shown in the literature.

Idea: The absolute value of the complex reflection coefficient is given by:

$$|R| = e^{-\frac{2\eta}{\cos\phi}} \prod_{j=0}^{q} \frac{(\cos\phi - \cos\phi_j)^2}{(\cos\phi + \cos\phi_j)^2}.$$

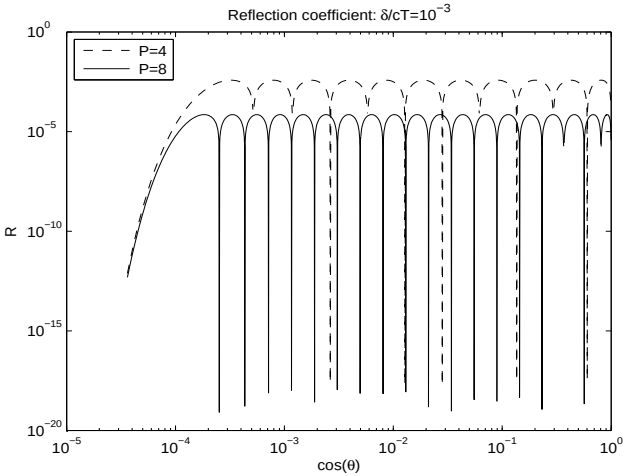
Given a tolerance ϵ we need only choose the interpolation nodes ϕ_j to make the product small for

$$\cos\phi > 2\eta \cdot \frac{1}{\ln\frac{1}{\epsilon}}.$$

For $\cos\phi_j = a^j$, $\cos\phi \in (a^{r+1}, a^r)$, $a = \left(\frac{2\eta}{\ln\frac{1}{\epsilon}}\right)^{1/q}$ we have:

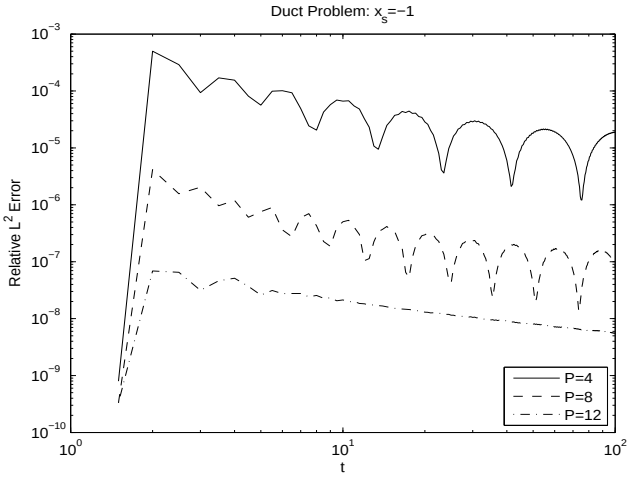
$$\begin{aligned} |R| &\leq \prod_{j=0}^{r} \left(\frac{a^j - a^{r+1}}{a^j + a^{r+1}}\right)^2 \cdot \prod_{j=r+1}^{q} \left(\frac{a^j - a^r}{a^j + a^r}\right)^2 \\ &\leq \prod_{j=1}^{q+1} \left(\frac{1-a^j}{1+a^j}\right)^2 \leq \prod_{j=1}^{q+1} e^{-4a^j} = e^{-4a\frac{1-a^{q+1}}{1-a}}. \end{aligned}$$

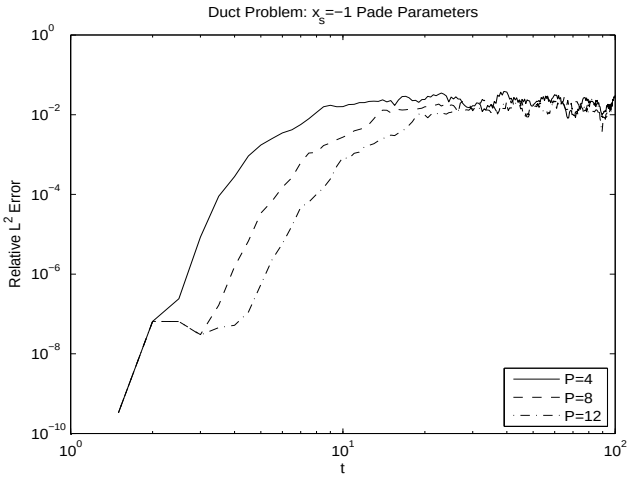
The tolerance is met if $a = 1 - O(\ln^{-1}\frac{1}{\epsilon})$ leading to the estimate of q. In practice we can use the Remez algorithm to compute optimal nodes. We emphasize that the reflection coefficient in this case directly leads to an error estimate.

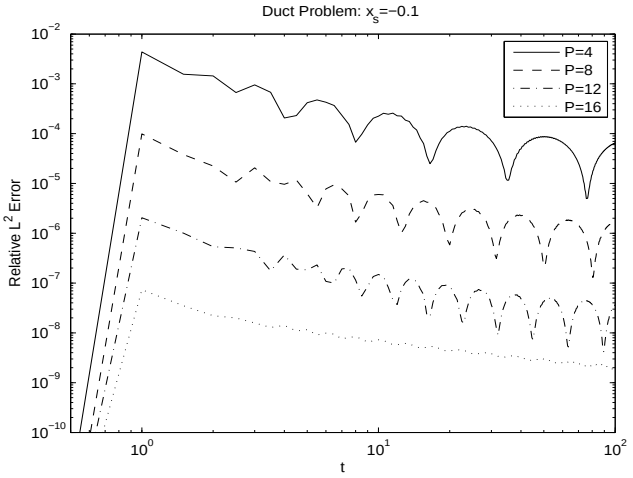

Reflection coefficient: δ/cT=10^−3
P=4
P=8
R
cos(θ)
10^0
10^−5
10^−10
10^−15
10^−20
10^−5
10^−4
10^−3
10^−2
10^−1
10^0

Here is a numerical example illustrating the improvement in accuracy obtained when the new approximations are included. (Kurt Stein is working on 3d implementations.) We solve the acoustic system in a unit duct with fields determined by point sources located a distance 1 from the inner boundary. The solution is well-resolved in space and time (Grid-stabilized 8th order spatial differencing combined 8th order Taylor time stepping.) We use $4-12$ terms in the boundary condition and compare results obtained using the new radiation conditions with the well-known Engquist-Majda sequence. Errors are computed using an accurate evaluation of an integral representation of the exact solution. We see that for $N_p = 12$ about 5 orders of magnitude in accuracy are gained. Also note that the long time error for the EM sequence is essentially independent of order as predicted by the analysis of Diaz and Joly (SIAP 2005). This is a direct consequence of their lack of accuracy for evanescent modes.

In the second experiment we consider the far more difficult case of a point source located a distance 0.1 from the inner boundary. Only a few more terms are needed to get excellent accuracy.


Duct Problem: x_s=−1
Relative L^2 Error
t
P=4
P=8
P=12


Duct Problem: x_s=−1 Pade Parameters
Relative L^2 Error
t
P=4
P=8
P=12


Duct Problem: x_s=−0.1
Relative L^2 Error
t
P=4
P=8
P=12
P=16
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{0}
10^{1}
10^{2}

In our next experiment we consider scattering from an array of perfectly conducting cylinders, computed using nodal DG discretizations with degree 9 polynomials. To use the new method in exterior domains which we bound by polygons we need **corner compatibility conditions** to connect the auxiliary variables on adjacent edges/faces. Such conditions have been studied for the Padé-based local boundary conditions by Collino (Waves02, 1993) and Vacus (Math. Comp., 2005) but their constructions are complex and of somewhat limited scope.

Here we construct corner closures by introducing multiply-indexed auxiliary variables at the corners which satisfy the recursions from each edge. We can combine the governing equations with the two sets of recursions to derive corner odes for the new variables. These in turn provide boundary conditions along the edges. (We can also view this construction as a standard corner layer using the PML analogy.)

$$\begin{aligned}
&\frac{\partial \psi_{k+1,j+1}}{\partial t} + \frac{\partial \psi_{k,j}}{\partial t} + \frac{\partial \psi_{k+1,j}}{\partial t} + \frac{\partial \psi_{k,j+1}}{\partial t} \\
&+S_j A \left(\psi_{k+1,j+1} - \psi_{k+1,j} + \psi_{k,j+1} - \psi_{k,j}\right) \\
&+S_k B \left(\psi_{k+1,j+1} - \psi_{k,j+1} + \psi_{k+1,j} - \psi_{k,j}\right) \quad = 0,
\end{aligned}$$

where

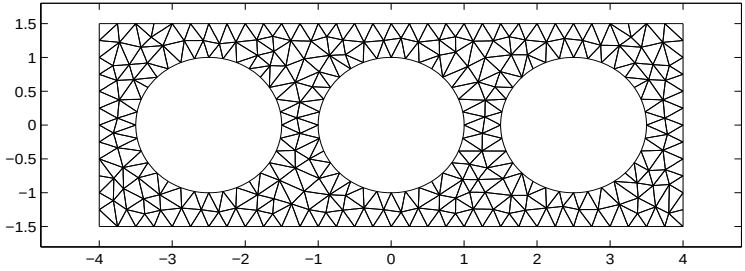
$$S_j w \equiv \cos\phi_j \frac{\partial w}{\partial t} + \frac{\sin^2\phi_j}{\cos\phi_j T} w.$$

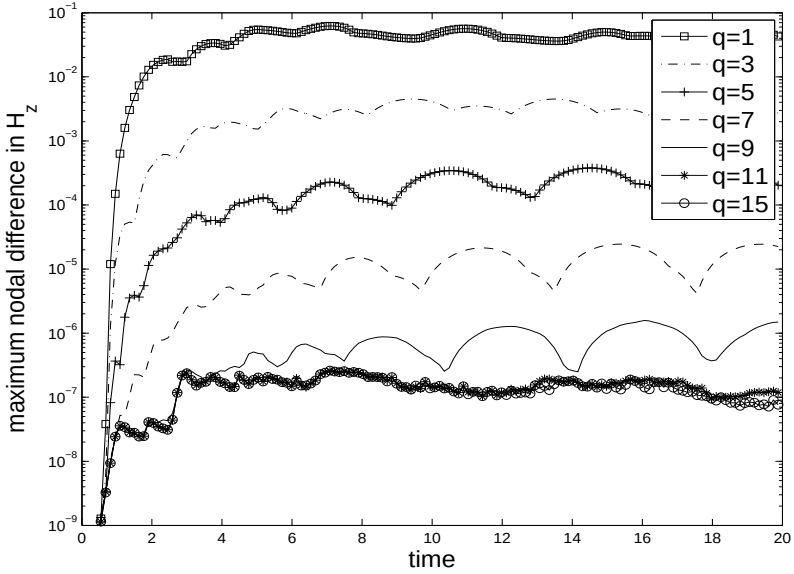
Truncate by:

$$\psi_{-,k,N_p+1} = \psi_{-,N_p+1,j} = 0,$$

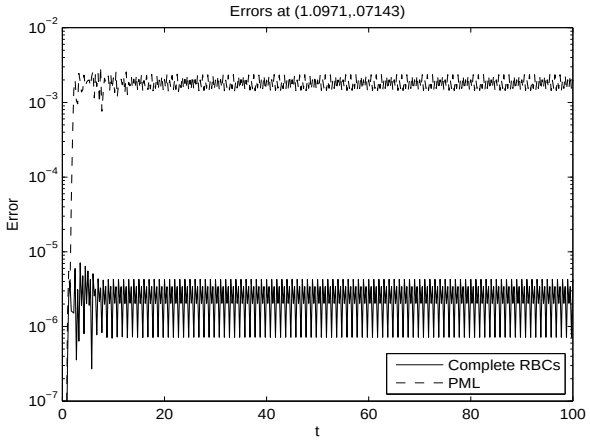
with $\frac{\partial \psi_{+,k,0}}{\partial t}$, $\frac{\partial \psi_{+,0,j}}{\partial t}$ computed from the edges. (Here the meanings of $+$ and $-$ are based on the local characteristics in the normal directions. The truncations are combined for $k = j = N_p + 1$.)

As yet we have no analysis of the corner closures but they appear to work in practice.


1.5
1
0.5
0
−0.5
−1
−1.5
−4
−3
−2
−1
0
1
2
3
4


maximum nodal difference in H_z
time
q=1
q=3
q=5
q=7
q=9
q=11
q=15

In the last example we will compare the performance of CRBC with corner closures to the performance of a standard PML with a uniform grid and a similar number of degrees of freedom. The problem is defined by a distributed oscillatory pressure source in $(-1,1)\times(-.1,.1)$ and the computational domain is $(-1.1,1.1)\times(-.2,.2)$. We use the PML formulation of Appelö, H. and Kreiss (SIAP 2006) and made some effort to optimize the parameters. Nonetheless the CRBCs are both cheaper and more accurate. Note that the CRBC parameters directly follow from setting $\eta = 10^{-3}$ and fixing the number of nodes - no tuning was needed!


Errors at (1.0971,.07143)
Error
t
Complete RBCs
PML

Open problems and work in progress:

Analysis and Improvement of Corner Conditions

As mentioned above we have no analysis of the corner closures. We are attempting to generalize Vacus' construction to CRBCs, but it is not entirely straightforward.

In addition it might be useful, particularly in 3 space dimensions, to simplify them. One possibility is to exploit a third way to develop RBCs - infinite elements. An interesting new though somewhat involved construction of infinite elements is based on the so-called pole condition (Hohage, Schmidt and Zschiedrich SIAP 2003, Ruprecht, Schädle, Schmidt and and Zschiedrich SISC 2008) - additional Laplace transform in space followed by an s-dependent Móbius transform coupled with Galerkin approximations on the resulting disk. I believe that CRBCs can be reproduced in their framework by choosing a different approximation scheme. They can treat some inhomogneous far fields and in particular construct corner closures with amny fewer variables (Nannen and Schädle, preprint).

Alternatively, one could adapt the analysis to construct optimal grids for standard (explicit!) PMLs which would lead to explicit corner closures.

General hyperbolic systems

Do good interpolation nodes always exist, and if so how can we find them?

If the sign of the imaginary part of $\mu_\pm$ does not change as the magnitute of $\Im s$ changes then the problem looks relatively straightforward. (This is the case where phase and group velocity point in the same direction.) What happens when they are mismatched?

For dispersion relations following from quadratic equations we can always effectively transform to an isotropic problem (completing the square!) Then CRBCs work directly. See Bécache, Givoli and H. (JCP 229 2010) and H., Bécache, Givoli and Stein (CICP to appear). What about more complicated cases, such as the problems in elasticity discussed in Bécache, Fauqueux and Joly (JCP 188 2003)?

In this case we cannot choose complex μ_j, $\bar{\mu}_j$ so that each term in the product defining the complex coefficient is 1. Purely real parameters, on the other hand, are very slowly convergent. It is interesting to note that the "damping layer" interpretation of the CRBCs in the case of purely real paramters is simply grid stretching. Grid stretching combined with damping is a popular technique in the engineering community. Appelö and Colonius (JCP 228 2009) show that this method can be quite accurate for elastic waves if high-order discretizations are used.

Looking at the reflection coefficient, it seems that a number of purely real parameters must be chosen to handle the case of purely tangential **phase velocity.** (For $\delta > 0$ we never have to worry about tangential group velocity.) However, it seems likely that choosing complex parameters in addition could enhance efficiency,

Variable coefficients

Coefficients which are variable tangential to the boundary as in stratified media can be treated in the same way as constant coefficient problems. One only needs to estimate the location of the eigenvalues which can be done (perhaps crudely) using the coefficients. Variation in the normal direction is more difficult. That said, Ehrhardt, Zheng and coworkers have recently completed some very interesting work on characterizing the exact conditions for periodic and (slowly) decaying potentials (JCP 2009, WIAS 2009). For decaying potentials the pole condition approach can also be applied.

The exact conditions formally follow from an operator Ricatti equation, which they treat by quadrature. It seems likely that this formulation could be combined with our interpolation scheme under some (as yet unformulated!) assumptions.

Alternatively, one could imagine combining the grid stretching method (Appeló and Colonius) mentioned earlier with some sort of goal-oriented grid adaption. The grid-stretching approach can be fit in with our interpolation scheme, and perhaps be improved by it. One can see that it will always work, so the main issue is efficiency. Another possibility is a multiscale filtering approach - applied to the Schrödinger equation with decaying potentials by Soffer and Stucchio (CPAM 2009).