Math 5316 - Spring 2009
Homework 3 - Problem 4
Solution

1) This is Theorem 4.2.15 in the text. Since $A - A_r = U \begin{pmatrix} \sigma_1 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix} V^T$

$\|A - A_r\|_2 = \sigma_{r+1}$

Suppose $\text{rank}(B) = r$. Then $\dim(\mathcal{N}(B)) = n - r$

Since the dimension of the span of $v_1, \ldots, v_{r+1}$ is $r+1$ we can conclude that there exists a vector $z_j, \|z_j\|_2 = 1$ in $\mathcal{N}(B) \cap \text{span } v_1, \ldots, v_{r+1}$

This follows from the fact that if the sum of the dimensions of two subspaces of $\mathbb{R}^n$ exceeds $n$, they must have a nontrivial intersection.

(To visualize this theorem think of 2 planes in $\mathbb{R}^3$.)

Since $z \in \text{span } v_1, \ldots, v_{r+1}$, we have

$Z = V(w)^3$ length $r+1$

$\|w\|_2 = 1$

$(A - B)z = Az = U \Sigma V^T (V(w))$

$= U (\Sigma) (w) = U (\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_{r+1} \end{pmatrix}) w$

Thus $\|A - B\| z \| = \| (\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_{r+1} \end{pmatrix}) w \| \geq \sigma_{r+1} \| w \| = \sigma_{r+1}$

Hence $\|A - B\| \geq \sigma_{r+1}$.
ii.) A singular nxn matrix has rank ≤ n-1. Thus the nearest singular matrix to A is An-1 and \( \| A - A_{n-1} \| = \sigma_n \).

From part i.,

Since \( \| A \| = \sigma_1 \),

\[
\frac{\| A - A_{n-1} \|}{\| A \|} = \frac{\sigma_n}{\sigma_1} = \frac{1}{x(A)}
\]