1. See Matlab script and graphs LUerr.pdf and LUsolerr.pdf. For the experiments we see that the LU factorization is stable in that \( \|LU - A\| \leq \varepsilon m \|A\| \). The solutions, too, were generally quite accurate, though a few showed the loss of some digits due to conditioning.

2. See hw2p2sol.txt. In this case the factorization is unstable as \( \|LU - A\| \approx 0.07 \|A\| \). As a result the solution is not very accurate. To understand the instability, let's replace \(-.99999\) by \(-1\) below the main diagonal. (I made it \(-.99999\) to make sure there were no ties in determining the pivot.) Then it is easy to carry out the elimination process to find \(U\):

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
-1 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & 0 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & 0 \\
\end{pmatrix}
\]

(Add row 1 to rows 2 \( \cdots \) \(n\))

(Add row 3 to rows 3 \( \cdots \) \(n\))

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & 0 \\
\end{pmatrix}
\]

etc.
Repeating this calculation we find:

\[
U = \begin{pmatrix}
1 & 0 \\
0 & \cdots
\end{pmatrix}
\]

Thus the entries of \( U \) grow exponentially in \( n \) \( \Rightarrow \) factorization is unstable.

3. See hw2p3sol.txt

4. See hw2p4sol.txt

5. Suppose \( A \) is column diagonally dominant. That is:

\[
|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad \text{for } i=1:n.
\]

In the first step of Gaussian elimination with partial pivoting we choose the pivot from \( \sum |a_{ii}| \). No pivoting is done in step 1.

Given the recursive construction of the elimination algorithm we are done if we prove that the \( (n-1) \times (n-1) \) matrix in rows 2:n and columns 2:n is still column diagonally dominant. Define this matrix as \( \hat{A}^{(i)} \), but for simplicity keep the original indices 2:n. Then we explicitly write down formulas for the entries of \( \hat{A}^{(i)} \). Notice we subtract \( \frac{a_{ij}}{a_{ii}} \) from row \( i \).
\[ a_{ji}^{(1)} = a_{ji} - \frac{a_{ji} a_{1i}}{a_{11}} \]

Now we check for column diagonal dominance:

\[ \sum_{j=2}^{n} |a_{ji}^{(1)}| = \sum_{j=2}^{n} \left| a_{ji} - \frac{a_{ji} a_{1i}}{a_{11}} \right| < |a_{ii}| - |a_{ii}| \]

\[ \leq \sum_{j=2}^{n} \left| a_{ji} \right| + \left| a_{1i} \right| - \frac{|a_{ii}||a_{1i}|}{|a_{11}|} \]

\[ = \sum_{j=2}^{n} \left| a_{ji} \right| - \frac{|a_{ii}||a_{1i}|}{|a_{11}|} \]

\[ < |a_{ii}| - \frac{|a_{ii}||a_{1i}|}{|a_{11}|} \]

\[ \leq |a_{ii} - \frac{a_{ii} a_{1i}}{a_{11}}| = |a_{ii}^{(1)}| \]

Thus we've shown:

\[ |a_{ii}^{(1)}| > \sum_{j=2}^{n} |a_{ji}^{(1)}| \]

\[ \Rightarrow A^{(1)} \text{ is column diagonally dominant.} \]

\[ \Rightarrow \text{No pivoting will be done. (See handwritten for illustration)} \]