Math 504 - CS 575 Final Exam

There are four problems of equal worth.

1. Show that if \( x \) and \( y \) are 2-vectors then:

\[
fl(x^T y) = \bar{x}^T \bar{y}, \quad \bar{x}_i = x_i(1 + \delta_i), \quad |\delta_i| \leq 2\epsilon_m.
\]

Here \( fl \) stands for floating point operations, \( \epsilon_m \) is the unit roundoff, and you may ignore \( O(\epsilon_m^2) \) terms. Does this result imply that the computed product has small relative error? Explain your answer.

2. Let \( A = B^T B \). Bound the 2-norm of \( B \) in terms of the 2-norm of \( A \). What is the implication of the result for the stability of the Cholesky factorization?

3. A Givens rotation is an orthogonal matrix, \( G(k, l, \theta) \), which is the identity matrix except for 4 elements; \( G_{kk} = G_{ll} = \cos \theta, G_{kl} = -G_{lk} = \sin \theta \). For example, the 6 × 6 Givens matrix \( G(2, 5, \theta) \) is given by:

\[
G(2, 5, \theta) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \cos \theta & 0 & 0 & \sin \theta & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -\sin \theta & 0 & 0 & \cos \theta & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Given a vector, \( y \), show how to apply a Givens rotation, \( G \), so that \((Gy)_l = 0 \). Is there any restriction on \( k \)? Briefly describe an efficient algorithm using Givens rotations to compute the QR factorization of an \( n \times n \) matrix, \( A \), which is upper triangular except for a single nonzero element, \( a_{n1} \neq 0 \). How does the work required scale with \( n \)? What is the structure of \( Q \)?

4. A matrix, \( A \), is called skew-symmetric if \( A^T = -A \). Briefly describe an efficient algorithm for computing the eigenvalues of such a matrix, explaining any simplifications which occur due to the skew-symmetry. What is the structure of the real Schur form?