A Demonstration of an
Additive Time Series Decomposition
Based on the SAS program Decomposition.sas

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In traditional time series analysis it is often assumed that a time series \( y_t \) can be **additively decomposed** into four components, namely, **trend**, **season**, **cycle**, and **irregular** components as in

\[
y_t = T_t + S_t + C_t + I_t
\]

where \( T_t \) represents the trend in \( y_t \) at time \( t \), \( S_t \) the seasonal effect at time \( t \), \( C_t \) the cyclical effect at time \( t \) and \( I_t \) the irregular effect at time \( t \).

To demonstrate this decomposition, consider the following characterizations of trend, cycle, seasonal, and irregular components that have been encoded in a SAS program entitled Decomposition.sas that is available on the class website.

\[
T_t = 100 + 4.0^*t \quad \text{(deterministic trend: intercept = 100, slope = 4)}
\]

\[
C_t = 50^*\cos(3.1416^*t/10) \quad \text{(deterministic cycle: amplitude = 50, Period = 20 months, phase = 0)}
\]

\[
S_t = \{ \text{fixed seasonal effects: -50, -25, 25, -25, -50, 50, 75, 50, 5, -25, -50, 20} \}
\]

\[
\text{(i.e. Jan. effect = -50, Feb. effect = -25, \ldots, Dec. effect = 20)}
\]

\[
I_t \rightarrow \text{NIID}(0,100)
\]

Therefore, the actual population model can be written as

\[
y_t = T_t + C_t + S_t + I_t
\]

\[
= 100 + 4.0^*t + 50\cos(0.31416^*t) - 50^*\gamma^1 - 25^*\gamma^2
\]

\[
+ 25^*\gamma^3 - 25^*\gamma^4 - 50^*\gamma^5 + 50^*\gamma^6
\]

\[
+ 75^*\gamma^7 + 50^*\gamma^8 + 5^*\gamma^9 - 25^*\gamma^{10}
\]

\[
- 50^*\gamma^{11} + 20^*\gamma^{12} + \epsilon_t.
\]

The above seasonal dummies are defined \( \gamma^i = 1 \) if the observation is in the \( i \)-th month and zero otherwise. Obviously the intercept of the month varies by month. For example, the intercept for all of the January months is \((100 - 50 = 50)\), the intercept for the February months is \((100 - 25 = 75)\), etc. The irregular component is represented by the unobserved error \( \epsilon_t \), which is normally and independently distributed with mean zero and variance of 100. The time index \( t \) ranges from 1 for the first observation and \( t = T \) for the last observation.

Each of these components is plotted in order in the following figures below:
Figure 1

Deterministic Trend Data
X=Time Y=Trend Series

Figure 2

Cycle with a=50, w = 2π/20 (period = 20 months), theta = 0
X=Time Y=Cyclical Series
In the following graphs we sum these components up as is intended in the additive decomposition (1).
And in Figure 8 below we have the representation of the fitted model obtained by Proc Nlin (a Nonlinear Least Squares procedure).
The fitted model is

\[ y_t = 51.74 + 3.98t + 50.76\cos(0.3140t - 0.0314) + 21.36 \cdot \text{gamma2} + 75.21 \cdot \text{gamma3} \\
+ 23.99 \cdot \text{gamma4} + 2.21 \cdot \text{gamma5} + 93.26 \cdot \text{gamma6} + 120.3 \cdot \text{gamma7} \\
+ 100.6 \cdot \text{gamma8} + 56.16 \cdot \text{gamma9} + 24.69 \cdot \text{gamma10} - 0.58 \cdot \text{gamma11} \\
+ 69.97 \cdot \text{gamma12} + \hat{\epsilon}_t \]

which closely matches the population parameters. Obviously, the January intercept is estimated as 51.74, the February intercept is 51.74 + 21.36 = 73.1, etc. The slope of the trend is 3.98 while the amplitude of the cycle is 50.76, the phase is -0.0314, and the period is

\[ p = 2\pi / w = 2 \cdot 3.1416 / 0.3140 = 20.01 \text{ months}. \]