Hasza & Fuller Test $H_0: \Delta, \Delta_s$

(1982), Annals of Statistics $H_1$: not $\Delta, \Delta_s$



TABLE 5.1

*Empirical percentiles for test statistics*

| | $m$ | $d=2$ | | | $d=4$ | | | $d=6$ | | | $d=12$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 |
| $\Phi^{(3)}_{n-3}$ | 10 | 2.52 | 3.24 | 5.09 | 2.46 | 3.06 | 4.49 | 2.42 | 3.00 | 4.35 | 2.37 | 2.93 | 4.23 |
| | 20 | 2.48 | 3.09 | 4.51 | 2.44 | 3.01 | 4.30 | 2.41 | 2.97 | 4.24 | 2.37 | 2.92 | 4.19 |
| | 50 | 2.45 | 3.02 | 4.29 | 2.44 | 2.99 | 4.22 | 2.41 | 2.96 | 4.19 | 2.37 | 2.92 | 4.17 |
| | $\infty$ | 2.44 | 2.98 | 4.20 | 2.44 | 2.98 | 4.19 | 2.41 | 2.96 | 4.17 | 2.37 | 2.92 | 4.16 |
| $\Phi^{(3)}_{n-d-4}$ | 10 | 8.33 | 10.08 | 14.86 | 8.95 | 10.42 | 13.72 | 10.09 | 11.53 | 14.58 | 13.49 | 15.13 | 18.47 |
| | 20 | 7.36 | 8.58 | 11.47 | 8.53 | 9.69 | 12.33 | 9.72 | 10.97 | 13.56 | 13.40 | 14.78 | 17.76 |
| | 50 | 6.97 | 7.93 | 9.92 | 8.26 | 9.28 | 11.36 | 9.54 | 10.68 | 12.86 | 13.24 | 14.55 | 17.46 |
| | $\infty$ | 6.67 | 7.50 | 9.17 | 8.04 | 8.96 | 10.90 | 9.36 | 10.39 | 12.43 | 13.16 | 14.41 | 16.93 |
| $\Phi^{(d+4)}_{n-d-4}$ | 10 | 5.31 | 6.37 | 9.36 | 4.32 | 4.96 | 6.35 | 3.99 | 4.46 | 5.56 | 3.54 | 3.84 | 4.47 |
| | 20 | 4.44 | 5.09 | 6.69 | 3.94 | 4.44 | 5.42 | 3.71 | 4.08 | 4.87 | 3.40 | 3.65 | 4.18 |
| | 50 | 4.06 | 4.54 | 5.63 | 3.73 | 4.12 | 4.96 | 3.57 | 3.90 | 4.59 | 3.29 | 3.52 | 4.05 |
| | $\infty$ | 3.81 | 4.22 | 5.05 | 3.60 | 3.93 | 4.62 | 3.45 | 3.73 | 4.35 | 3.22 | 3.44 | 3.88 |
| $\Phi^{(2)}_{n-2}$ | 10 | 2.61 | 3.46 | 5.76 | 2.58 | 3.36 | 5.20 | 2.53 | 3.28 | 5.04 | 2.46 | 3.19 | 4.90 |
| | 20 | 2.59 | 3.34 | 5.17 | 2.54 | 3.29 | 4.99 | 2.50 | 3.24 | 4.92 | 2.45 | 3.17 | 4.84 |
| | 50 | 2.55 | 3.27 | 4.99 | 2.53 | 3.26 | 4.92 | 2.49 | 3.22 | 4.88 | 2.45 | 3.16 | 4.83 |
| | $\infty$ | 2.52 | 3.24 | 4.93 | 2.52 | 3.24 | 4.90 | 2.49 | 3.20 | 4.87 | 2.45 | 3.15 | 4.82 |
| $\Phi^{(2)}_{n-d-3}$ | 10 | 7.94 | 9.77 | 14.72 | 9.84 | 11.68 | 15.87 | 11.75 | 13.68 | 17.91 | 17.37 | 19.53 | 24.22 |
| | 20 | 7.43 | 8.89 | 12.03 | 9.63 | 11.12 | 14.39 | 11.57 | 13.24 | 16.81 | 17.29 | 19.24 | 23.65 |
| | 50 | 7.23 | 8.44 | 11.30 | 9.41 | 10.78 | 13.82 | 11.46 | 12.93 | 16.30 | 17.21 | 19.15 | 23.18 |
| | $\infty$ | 7.05 | 8.16 | 10.48 | 9.28 | 10.59 | 13.26 | 11.36 | 12.82 | 15.72 | 17.20 | 19.08 | 22.61 |
| $\Phi^{(d+3)}_{n-d-3}$ | 10 | 4.39 | 5.30 | 7.64 | 3.87 | 4.47 | 5.84 | 3.64 | 4.11 | 5.15 | 3.37 | 3.68 | 4.31 |
| | 20 | 3.87 | 4.50 | 6.00 | 3.59 | 4.05 | 5.08 | 3.44 | 3.84 | 4.61 | 3.24 | 3.50 | 4.04 |
| | 50 | 3.61 | 4.14 | 5.26 | 3.45 | 3.86 | 4.74 | 3.34 | 3.68 | 4.38 | 3.16 | 3.40 | 3.90 |
| | $\infty$ | 3.45 | 3.92 | 4.88 | 3.35 | 3.73 | 4.48 | 3.26 | 3.58 | 4.23 | 3.11 | 3.34 | 3.80 |

THEOREM 4.1. *Let $Y_t$ satisfy model* (4.1) *with $e_t$ that are* iid$(0, \sigma^2)$. *Let* **H**, **h** *be as defined in Corollary* 3.1. *Then under $H_0: \beta' = (1, 0, 1)$*

(i) $$\{n^2(\hat{\beta}_1 - 1), n\hat{\beta}_2, n(\hat{\beta}_3 - 1)\}' \rightarrow_{\mathscr{L}} c\mathbf{H}^{-1}\mathbf{h},$$

(ii) $$n^{1/2}(\hat{\theta} - \theta) \rightarrow_{\mathscr{L}} N_p(0, \Gamma^{-1}\sigma^2),$$

*where* $(\Gamma)_{ij} = \lim_{t\to\infty} \mathrm{Cov}(X_t, X_{t+|i-j|})$.

PROOF. Define

$$W^{\dagger}_t = \sum_{j=1}^{t} e_t,\ Z^{\dagger}_t = \sum_{j=1}^{[t/d]} e_{t-dj},\ Y^{\dagger}_t = \sum_{j=1}^{t} Z^{\dagger}_j.$$

It is not difficult to show that the asymptotic properties of $(\sum_{t=1}^{n} \psi_t\psi_t')^{-1}(\sum_{t=1}^{n} \psi_t e_t)$ are not affected by replacing $Y_{t-1}$, $W_{t-1}$, and $Z_{t-d}$ by $c^{-1}Y^{\dagger}_{t-1}$, $c^{-1}W^{\dagger}_{t-1}$, and $c^{-1}Z^{\dagger}_{t-d}$ respectively. Furthermore

$$\sum_{t=1}^{n} Y_{t-1}X_{t-i} = O_p(n^2), \qquad \sum_{t=1}^{n} W_{t-1}X_{t-1} = O_p(n), \qquad \sum_{t=1}^{n} Z_{t-d}X_{t-i} = O_p(n),$$
$$i = 1, 2, \cdots, p.$$

The result then follows from Theorem 3.1 and the well-known asymptotic distributional theory for stationary autoregressive processes. □

Now let $\Phi^{(3)}_{n-p-3}$ denote the test statistic for testing $H_0: \beta' = (1, 0, 1)$ analogous to the usual $F_{3,n-p-3}$ test statistic in a fixed normal regression model.

(1984) Journal of American Statistical Association

Dickey, Hasza, and Fuller Test

$H_0$: $\Delta_s$

$H_1$: not $\Delta_s$



*Table 5. Percentiles for $\tau_{\mu d}{}^*$, the Studentized Test for the Single Mean Model*

| | | Probability of a Smaller Value | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | n = md | .01 | .025 | .05 | .10 | .50 | .90 | .95 | .975 | .99 |
| | 20 | −3.54 | −3.08 | −2.72 | −2.32 | −.97 | .49 | .92 | 1.31 | 1.76 |
| | 30 | −3.44 | −3.02 | −2.69 | −2.31 | −1.01 | .46 | .88 | 1.25 | 1.68 |
| | 40 | −3.40 | −3.00 | −2.68 | −2.31 | −1.02 | .44 | .86 | 1.22 | 1.65 |
| d = 2 | 100 | −3.31 | −2.95 | −2.65 | −2.31 | −1.05 | .41 | .83 | 1.19 | 1.61 |
| | 200 | −3.28 | −2.93 | −2.64 | −2.31 | −1.05 | .40 | .83 | 1.19 | 1.60 |
| | 400 | −3.27 | −2.93 | −2.64 | −2.31 | −1.06 | .40 | .82 | 1.18 | 1.59 |
| | ∞ | −3.25 | −2.92 | −2.63 | −2.31 | −1.06 | .40 | .82 | 1.18 | 1.59 |
| | 40 | −3.14 | −2.73 | −2.38 | −2.00 | −.60 | .80 | 1.19 | 1.54 | 1.94 |
| | 60 | −3.11 | −2.71 | −2.38 | −2.00 | −.63 | .76 | 1.15 | 1.49 | 1.89 |
| | 80 | −3.09 | −2.71 | −2.38 | −2.01 | −.64 | .74 | 1.13 | 1.47 | 1.87 |
| d = 4 | 200 | −3.07 | −2.70 | −2.38 | −2.02 | −.66 | .72 | 1.11 | 1.46 | 1.86 |
| | 400 | −3.06 | −2.70 | −2.38 | −2.02 | −.67 | .72 | 1.11 | 1.45 | 1.85 |
| | 800 | −3.06 | −2.70 | −2.38 | −2.02 | −.67 | .72 | 1.10 | 1.45 | 1.85 |
| | ∞ | −3.05 | −2.70 | −2.38 | −2.03 | −.67 | .72 | 1.10 | 1.45 | 1.85 |
| | 120 | −2.73 | −2.33 | −2.01 | −1.65 | −.31 | 1.02 | 1.38 | 1.71 | 2.08 |
| | 180 | −2.73 | −2.35 | −2.02 | −1.66 | −.33 | .99 | 1.35 | 1.68 | 2.06 |
| | 240 | −2.73 | −2.36 | −2.02 | −1.66 | −.34 | .98 | 1.34 | 1.67 | 2.05 |
| d = 12 | 600 | −2.73 | −2.37 | −2.04 | −1.66 | −.35 | .97 | 1.34 | 1.66 | 2.03 |
| | 1,200 | −2.73 | −2.37 | −2.05 | −1.66 | −.35 | .96 | 1.33 | 1.65 | 2.02 |
| | 2,400 | −2.74 | −2.37 | −2.06 | −1.66 | −.36 | .96 | 1.33 | 1.65 | 2.02 |
| | ∞ | −2.74 | −2.37 | −2.06 | −1.66 | −.36 | .96 | 1.33 | 1.65 | 2.01 |

would obtain by regressing $Z_t - Z_{t-d} = \dot{Z}_t$ on $Z_{t-d}$, where $Z_t = Y_t - \theta_1 Y_{t-1} - \cdots - \theta_p Y_{t-p}$. The estimators $\bar{\theta}_i$, obtained by adding the estimates of $\theta_i - \hat{\theta}_i$ to $\hat{\theta}_i$, have the same asymptotic distribution as the coefficients in a regression of $\dot{Y}_t$ on $\dot{Y}_{t-1}, \dot{Y}_{t-2}, \ldots, \dot{Y}_{t-p}$.

The proof of Theorem 5 is given in Appendix B. Theorem 5 implies that the tabulated limit percentiles for estimators in model (1.1) are applicable in the multiplicative model for large sample sizes.

The extension of Theorem 5 to estimators with seasonal means or a single mean is immediate. Let

$$y_t = Y_t - \sum_{i=1}^{d} \delta_{it}\bar{\mu}_i. \tag{6.3}$$

Replacing $Y_t$ by $y_t$ in the two-step estimation procedure results in the regression of $e_t(1, \hat{\theta})$ on

$$(1 - \hat{\theta}_1 B - \hat{\theta}_2 B^2 - \cdots - \hat{\theta}_p B^p) y_{t-d},$$

$$\dot{Y}_{t-1}, \dot{Y}_{t-2}, \ldots, \dot{Y}_{t-d}.$$

*Table 6. Percentiles for $n(\hat{\alpha}_{\mu d} - 1)$, the Ordinary Regression Coefficient for the Seasonal Means Model*

| | | Probability of a Smaller Value | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | n = md | .01 | .025 | .05 | .10 | .50 | .90 | .95 | .975 | .99 |
| | 20 | −18.98 | −16.75 | −14.96 | −12.78 | −6.48 | −1.94 | −.83 | .10 | 1.16 |
| | 30 | −20.89 | −18.19 | −16.01 | −13.62 | −6.76 | −2.11 | −1.01 | −.11 | .92 |
| | 40 | −22.02 | −19.03 | −16.64 | −14.09 | −6.91 | −2.18 | −1.10 | −.21 | .79 |
| d = 2 | 100 | −24.32 | −20.76 | −17.95 | −15.05 | −7.20 | −2.31 | −1.25 | −.40 | .55 |
| | 200 | −25.17 | −21.39 | −18.44 | −15.40 | −7.30 | −2.35 | −1.30 | −.46 | .46 |
| | 400 | −25.62 | −21.72 | −18.69 | −15.58 | −7.35 | −2.37 | −1.33 | −.49 | .42 |
| | ∞ | −26.07 | −22.06 | −18.95 | −15.76 | −7.41 | −2.39 | −1.35 | −.52 | .38 |
| | 40 | −28.78 | −25.59 | −23.10 | −20.40 | −11.89 | −5.40 | −3.90 | −2.51 | −1.05 |
| | 60 | −30.63 | −27.18 | −24.49 | −21.42 | −12.33 | −5.73 | −4.15 | −2.81 | −1.33 |
| | 80 | −31.79 | −28.12 | −25.27 | −22.00 | −12.58 | −5.87 | −4.28 | −2.96 | −1.47 |
| d = 4 | 200 | −34.26 | −30.03 | −26.78 | −23.15 | −13.06 | −6.09 | −4.51 | −3.20 | −1.74 |
| | 400 | −35.20 | −30.73 | −27.32 | −23.56 | −13.23 | −6.15 | −4.59 | −3.28 | −1.83 |
| | 800 | −35.69 | −31.10 | −27.60 | −23.78 | −13.32 | −6.18 | −4.63 | −3.32 | −1.87 |
| | ∞ | −36.19 | −31.47 | −27.88 | −24.00 | −13.41 | −6.21 | −4.67 | −3.35 | −1.92 |
| | 120 | −60.72 | −55.63 | −51.22 | −46.98 | −33.65 | −22.13 | −19.46 | −17.09 | −14.25 |
| | 180 | −63.40 | −57.83 | −53.57 | −49.14 | −34.95 | −23.06 | −20.00 | −17.44 | −14.57 |
| | 240 | −64.90 | −59.21 | −54.89 | −50.28 | −35.57 | −23.48 | −20.31 | −17.70 | −14.86 |
| d = 12 | 600 | −67.87 | −62.16 | −57.52 | −52.44 | −36.65 | −24.16 | −20.92 | −18.34 | −15.57 |
| | 1,200 | −68.93 | −63.29 | −58.47 | −53.19 | −36.99 | −24.37 | −21.14 | −18.60 | −15.87 |
| | 2,400 | −69.48 | −63.87 | −58.95 | −53.57 | −37.16 | −24.47 | −21.26 | −18.74 | −16.03 |
| | ∞ | −70.04 | −64.48 | −59.45 | −53.95 | −37.33 | −24.56 | −21.37 | −18.88 | −16.20 |