Model 1: Independent Box-Jenkins Time Series

Assume that $x_t$ and $y_t$ are stationary, that is, they are both I(0) and do not need to be differenced. Furthermore assume that $x_t$ and $y_t$ are independent in that $x$ does not Granger-cause $y$ $(x \mapsto y)$ and $y$ does not Granger-cause $x$ $(y \mapsto x)$. Model 1 is represented by

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \cdots + \alpha_p y_{t-p} + a_t - \tau_1 a_{t-1} - \cdots - \tau_q a_{t-q} \tag{2.5}$$

$$x_t = \phi_0 + \phi_1 x_{t-1} + \cdots + \phi_r x_{t-r} + v_t - \theta_1 v_{t-1} - \cdots - \theta_s v_{t-s}, \tag{2.6}$$

where $a_t$ and $v_t$ are independent white noise error terms. That is, $y_t$ follows an ARMA(p,q) Box-Jenkins process and $x_t$ follows an ARMA(r,s) Box-Jenkins process both of which are independent of each other. In the case that either $y_t$ or $x_t$ is I(1) or both are I(1) but not cointegrated, the $y_t$'s and/or $x_t$'s in the above equations (2.5) and/or (2.6) are replaced by their stationary forms, i.e. $\Delta y_t$ and/or $\Delta x_t$.

Model 2: Transfer Function Model

Assume that $x_t$ and $y_t$ are stationary, that is they are both I(0), and furthermore that $x_t$ Granger-causes $y_t$ ($x \rightarrow y$) but $y_t$ does not Granger-cause $x_t$ ($y \mapsto x$). That is, there is one-way causality from $x$ to $y$ but not the reverse. Then Model 2 is taken to be the classic Transfer Function model (Box and Jenkins (1970,76)):

$$y_t = \mu_x + \frac{\omega(B)}{\delta(B)} x_{t-b} + \frac{\theta(B)}{\phi(B)} a_t \tag{2.7}$$

$$x_t = \mu_y + \frac{\tau(B)}{\pi(B)} v_t, \tag{2.8}$$

where $a_t$ and $v_t$ are independent white noise error terms and the various backshift polynomials follow the forms

$$\omega(B) = 1 - \omega_1 B - \omega_2 B^2 - \cdots - \omega_r B^r$$

$$\delta(B) = 1 - \delta_1 B - \delta_2 B^2 - \cdots - \delta_s B^s$$

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q$$

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$$

$$\tau(B) = 1 - \tau_1 B - \tau_2 B^2 - \ldots - \tau_m B^m$$

$$\pi(B) = 1 - \pi_1 B - \pi_2 B^2 - \cdots - \pi_n B^n .$$

Equation (2.7) represents the systematic dynamics equation of the Transfer Function model while equation (2.8) represents the exogenous (leading indicator) variable equation. Obviously equation (2.7) is a rational distributed lag model with ARMA errors and equation (2.8) is a Box-Jenkins ARMA(m,n) model for the exogenous variable $x_t$.

If instead $y_t$ Granger-causes $x_t$ ( $y \rightarrow x$ ) but $x_t$ does not Granger-cause $y_t$ ( $x \mapsto y$ ) then the roles of $x$ and $y$ should be reversed in the above equations (2.7) and (2.8). Of course, should either $y_t$ or $x_t$ be I(1), or both are I(1) but not cointegrated, the $y_t$'s and/or $x_t$'s in the equations (2.7) and/or (2.8) should be replaced by their stationary forms, i.e. $\Delta y_t$ and/or $\Delta x_t$ .

<u>Model 3: Equal Lag-length VAR</u>

Assume that $x_t$ and $y_t$ are stationary ( $x_t$ is I(0) and $y_t$ is I(0)). Furthermore assume that $x_t$ and $y_t$ are two-way causal in the Granger-sense, i.e. $x \rightarrow y$ and $y \rightarrow x$ . Then Model 3 is represented by

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \cdots + \alpha_\ell y_{t-\ell} + \beta_1 x_{t-1} + \cdots + \beta_\ell x_{t-\ell} + e_{t1} \qquad (2.9)$$

$$x_t = \theta_0 + \theta_1 x_{t-1} + \cdots + \theta_\ell x_{t-\ell} + \tau_1 y_{t-1} + \cdots + \tau_\ell y_{t-\ell} + e_{t2} \qquad (2.10)$$

This model is the classic equal-lag length vector autoregression (VAR) of Sims (1980). Again, if either $y_t$ or $x_t$ is I(1) or both are I(1) but not cointegrated, the $y_t$'s and/or the $x_t$'s in the above equations (2.9) and/or (2.10) should be replaced by their stationary forms, i.e. $\Delta y_t, \Delta x_t$.

<u>Model 4: Error Correction Model</u>

Assume that $x_t$ and $y_t$ are both I(1) and that they are cointegrated with cointegrating relationship $z_t = \beta_0 + \beta_1 y_t + \beta_2 x_t + \beta_3 t$, where $z_t$ is an I(0) process with zero mean. (The most common case assumes $\beta_3 = 0$ and therefore that the time trend is absent from the cointegrating relationship.)

The most general Error Correction Model (ECM) is that of Johansen (1995, pp. 80 - 84):

$$\Delta y_t = \alpha_0 + \alpha_1 \Delta y_{t-1} + \cdots + \alpha_\ell \Delta y_{t-\ell} + \theta_1 \Delta x_{t-1} + \cdots + \theta_\ell \Delta x_{t-\ell}$$

$$+ \delta_1(\beta_0 + \beta_1 y_{t-1} + \beta_2 x_{t-1} + \beta_3 (t-1)) + \gamma_1 t + \varepsilon_{t1} \qquad (2.11)$$

$$\Delta x_t = \pi_0 + \pi_1 \Delta x_{t-1} + \cdots + \pi_\ell \Delta x_{t-\ell} + \varphi_1 \Delta y_{t-1} + \cdots + \varphi_\ell \Delta y_{t-\ell}$$

$$+ \delta_2(\beta_0 + \beta_1 y_{t-1} + \beta_2 x_{t-1} + \beta_3 (t-1)) + \gamma_2 t + \varepsilon_{t2}\,. \qquad (2.12)$$

The ECM of equations (2.11) and (2.12) is quite general and gives rise to five nested models. These models are:

a. Series $y_t$ and $x_t$ have no deterministic trends and the cointegrating relationship has no intercept (i.e. $\alpha_0 = \pi_0 = \beta_0 = \beta_3 = \gamma_1 = \gamma_2 = 0$).
b. Series $y_t$ and $x_t$ have no deterministic trends but the cointegrating relationship has an intercept (i.e. $\beta_0 \neq 0$ but $\alpha_0 = \pi_0 = \beta_3 = \gamma_1 = \gamma_2 = 0$).
c. Series $y_t$ and $x_t$ have linear trends but the cointegrating relationship has only an intercept (i.e. $\alpha_0, \pi_0$, and $\beta_0 \neq 0$ but $\beta_3 = \gamma_1 = \gamma_2 = 0$).
d. Both $y_t$ and $x_t$ have linear trends and the cointegrating relationship has a deterministic trend as well (i.e. $\alpha_0, \pi_0, \beta_0$, and $\beta_3 \neq 0$ but $\gamma_1 = \gamma_2 = 0$).
e. Series $y_t$ and $x_t$ have quadratic trends while the cointegrating relationship has a linear deterministic trend (i.e. $\alpha_0, \pi_0, \beta_0, \beta_3, \gamma_1$, and $\gamma_2 \neq 0$).

These five cases are nested from the most restrictive, case a., to the least restrictive, case e. These cases can be distinguished by examining a series of likelihood ratio tests as provided by the computer program EVIEWS (1997, Version 3, pp. 507-08). Also see Johansen (1995, pp. 80-84). Each of these cases is represented in the Monte Carlo data sets I provide to the students.