Lecture 6

Mean and Variance of $\hat{\beta}_1$

Sampling Distribution of $\hat{\beta}_1$

Note that

$$\hat{\beta}_1 = \sum_1^N w_i y_i$$

where
$$w_i = \frac{X_i - \bar{X}}{\sum_1^N (X_i - \bar{X})^2} .$$

The following properties hold for the w_i:

1) $\sum_1^N w_i = 0$

2) $\sum_1^N w_i X_i = 1$.

Property 1: $E(\hat{\beta}_1) = \beta_1$. ($\hat{\beta}_1$ is an unbiased estimator of β_1)

Proof:

$$E(\hat{\beta}_1) = E\left(\sum_1^N w_i y_i\right) = \sum_1^N E(w_i y_i)$$

$$= \sum_1^N w_i E(y_i) = \sum_1^N w_i E(\beta_0 + \beta_1 X_i + u_i)$$

$$= \sum_1^N w_i(\beta_0 + \beta_1 X_i) = \beta_0 \sum_1^N w_i + \beta_1 \sum_1^N w_i X_i$$

$$= \beta_1 \ .$$

<u>Property 2</u> : $\text{Var}(\hat{\beta}_1) = \dfrac{\sigma^2}{\sum_1^N (X_i - \bar{X})^2}$.

Proof:

Note: $$\hat{\beta}_1 = \sum_1^N w_i y_i = \sum_1^N w_i(\beta_0 + \beta_1 X_i + u_i)$$

$$= \beta_0 \sum_1^N w_i + \beta_1 \sum_1^N w_i X_i + \sum_1^N w_i u_i$$

$$= \beta_1 + \sum_1^N w_i u_i$$

$$\therefore \quad \hat{\beta}_1 - \beta_1 = \sum_1^N w_i u_i \ .$$

Now

$$\text{Var}(\hat{\beta}_1) = E(\hat{\beta}_1 - E(\hat{\beta}_1))^2 = E(\hat{\beta}_1 - \beta_1)^2$$

$$= E\left(\sum_1^N w_i u_i\right)^2$$

$$= E\left[\sum_1^N w_i^2 u_i^2 + \sum_{i \neq j} w_i w_j u_i u_j\right]$$

$$= \sum_1^N w_i^2 E(u_i^2) + \sum_{i \neq j} w_i w_j E(u_i u_j)$$

$$= \sum_1^N w_i^2 \sigma^2 + 0 = \sum_1^N w_i^2 \sigma^2 = \sigma^2 \sum_1^N \left[\frac{(X_i - \bar{X})}{\sum_1^N (X_i - \bar{X})^2} \right]^2$$

$$= \frac{\sigma^2}{\left(\sum_1^N (X_i - \bar{X})^2 \right)^2} \cdot \sum_1^N (X_i - \bar{X})^2$$

$$= \frac{\sigma^2}{\sum_1^N (X_i - \bar{X})^2} .$$

<u>Property 3</u>: <u>The sampling distribution</u> of $\hat{\beta}_1$

is $N\left(\beta_1 , \frac{\sigma^2}{\sum_1^N (X_i - \bar{X})^2} \right)$.

Proof:

If SLR.6 holds ($u_i \overset{iid}{\sim} N(0, \sigma^2)$) then

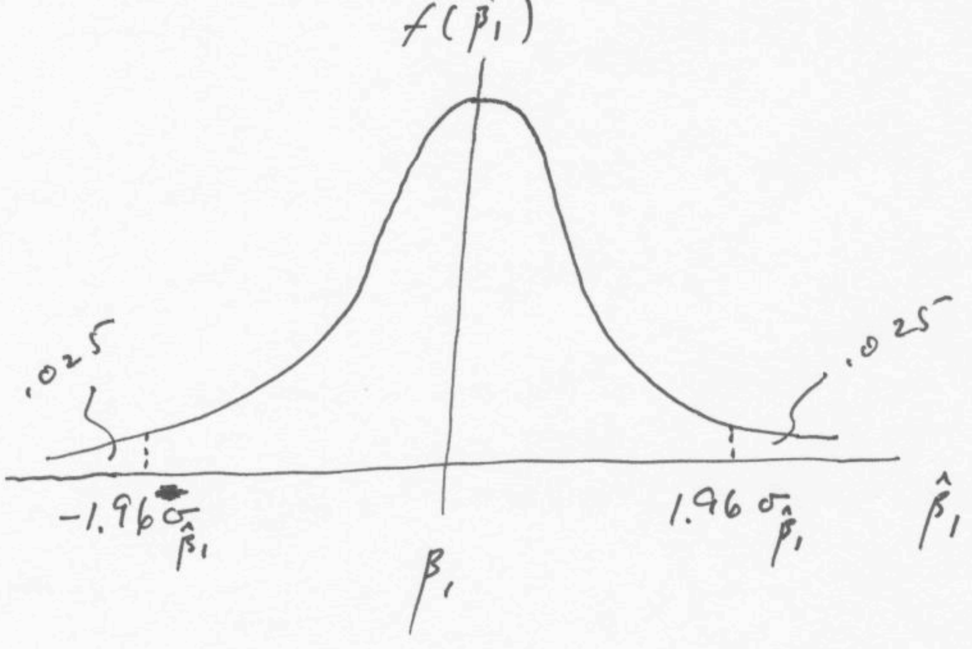
$$\hat{\beta}_1 = \sum_1^N w_i y_i = \beta_1 + \sum_1^N w_i u_i .$$

But $\sum_1^N w_i u_i$ is a linear combination of normal random variables and by the reproductive property of the normal distribution (i.e. a linear combination of normal random variables is again a normal random variable) is also normally distributed with mean zero and $Var\left(\sum_1^N w_i u_i\right) = \frac{\sigma^2}{\sum_1^N (X_i - \bar{X})^2}$.

Now $\hat{\beta}_1$ is equal to β_1 plus a normal random variable with mean 0 and variance $\frac{\sigma^2}{\sum_1^N (X_i - \bar{X})^2}$,

Thus $\hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma^2}{\sum_1^N (X_i - \bar{X})^2}\right)$.

Thus, under SLR.6, the sampling distribution of $\hat{\beta}_1$ is pictured below (assuming $N > 120$)



Note: The less the variation in the X_i around $\bar{X}$, the greater the variance of the sampling distribution of $\hat{\beta}_1$.

$$var(\hat{\beta}_1) = \frac{\sigma^2}{\sum_1^N (X_i - \bar{X})^2}.$$ If there is no variation in the X_i we have $Var(\hat{\beta}_1) = \infty$.

In a similar manner it can be shown that

Property 4: $E(\hat{\beta}_0) = \beta_0$ (unbiased)

Property 5: $$Var(\hat{\beta}_0) = \frac{\sum_1^N X_i^2}{N \sum_1^N (X_i - \bar{X})^2} \cdot \sigma^2$$

Property 6: Assuming SLR.6

$$\hat{\beta}_0 \sim N\left(\beta_0, \frac{\sum_1^N X_i^2}{N \sum_1^N (X_i - \bar{X})^2}\right).$$

Property 7: $$cov(\hat{\beta}_0, \hat{\beta}_1) = -\bar{X}\left(\frac{\sigma^2}{\sum_1^N (X_i - \bar{X})^2}\right)$$