8.2.8

$\dot{x} = x[x(1-x) - y]$

$\dot{y} = y(x-a)$

EP: $x[x(1-x)-y] = 0$

$x = 0 \quad y = x(1-x)$

$y(x-a) = 0$

If $x=0 \Rightarrow y=0$

If $y = x(1-x)$

$x(1-x)(x-a) = 0$

$x = 0, 1, a$

$\Rightarrow (x,y) = (0,0)\ (1,0)$

$(a, a(1-a))$

$Df = \begin{pmatrix} 2x - 3x^2 - y & -x \\ y & x-a \end{pmatrix}$

$Df|_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix} \quad \lambda = 0, -a$

Singular case ($\lambda=0$)

$Df|_{(1,0)} = \begin{pmatrix} -1 & -1 \\ 0 & 1-a \end{pmatrix} \quad \lambda = -1, (1-a)$

$a<1$: Saddle

$a>1$: A. Stable Node

$Df|_{(a,a(1-a))} = 0 \Rightarrow$

$\lambda^2 - \lambda a(1-2a) + a^2(1-a) = 0$

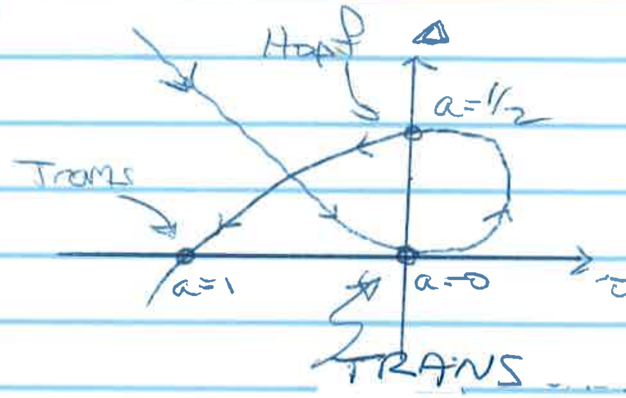
When $a=0$: $\tau=0$ & $\Delta=0$

Weird bif w/ $(x,y)=(0,0)$

When $a=1/2$: $\tau=0 \quad \Delta>0$ HOPF

When $a=1$: $\tau<0 \quad \Delta=0$ Bif with $(1,0)$

I used the computer to plot $(\tau, \Delta) = (a(1-2a), a^2(1-a))$ as a is increased.

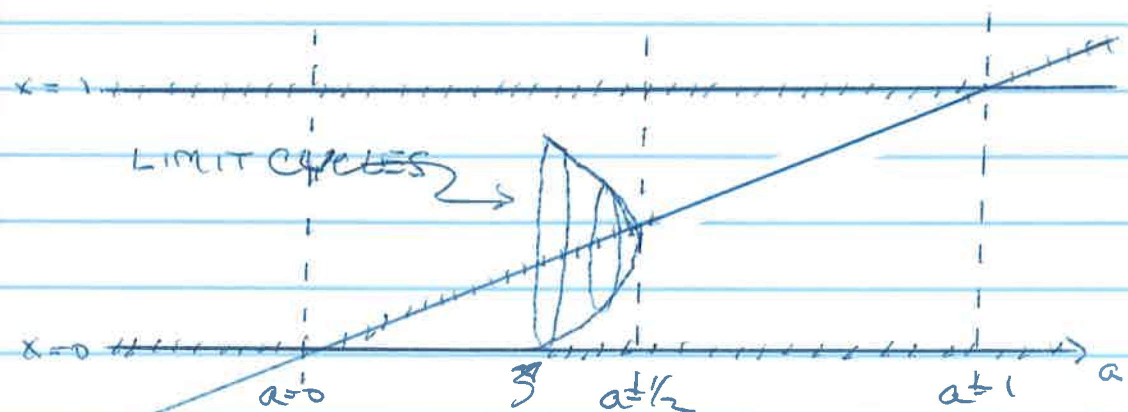


The main thing to observe is that changes, whatever they may be, occur at $a=0, \frac{1}{2}$ and 1. We suspect a Hopf when $a=\frac{1}{2}$

To determine if the Hopf is Super vs Subcrit, use PPLANE. Turns out it is Supercrit as a is DECREASED below ½. At $a = \frac{1}{2}$

$$\lambda = \pm\left[-\left(\tfrac{1}{4}\right)\left(1-\tfrac{1}{2}\right)\right]^{1/2} = \pm i\sqrt{\tfrac{1}{8}} \Rightarrow \omega \approx \sqrt{\tfrac{1}{8}}$$

Putting all the pieces together gives the following bif. diagram



$a \approx 0.33$

As a decreases the limit cycle collides w/ $x=0$, forming a Homoclinic orbit. For $a < 0.33$ the L.C is gone

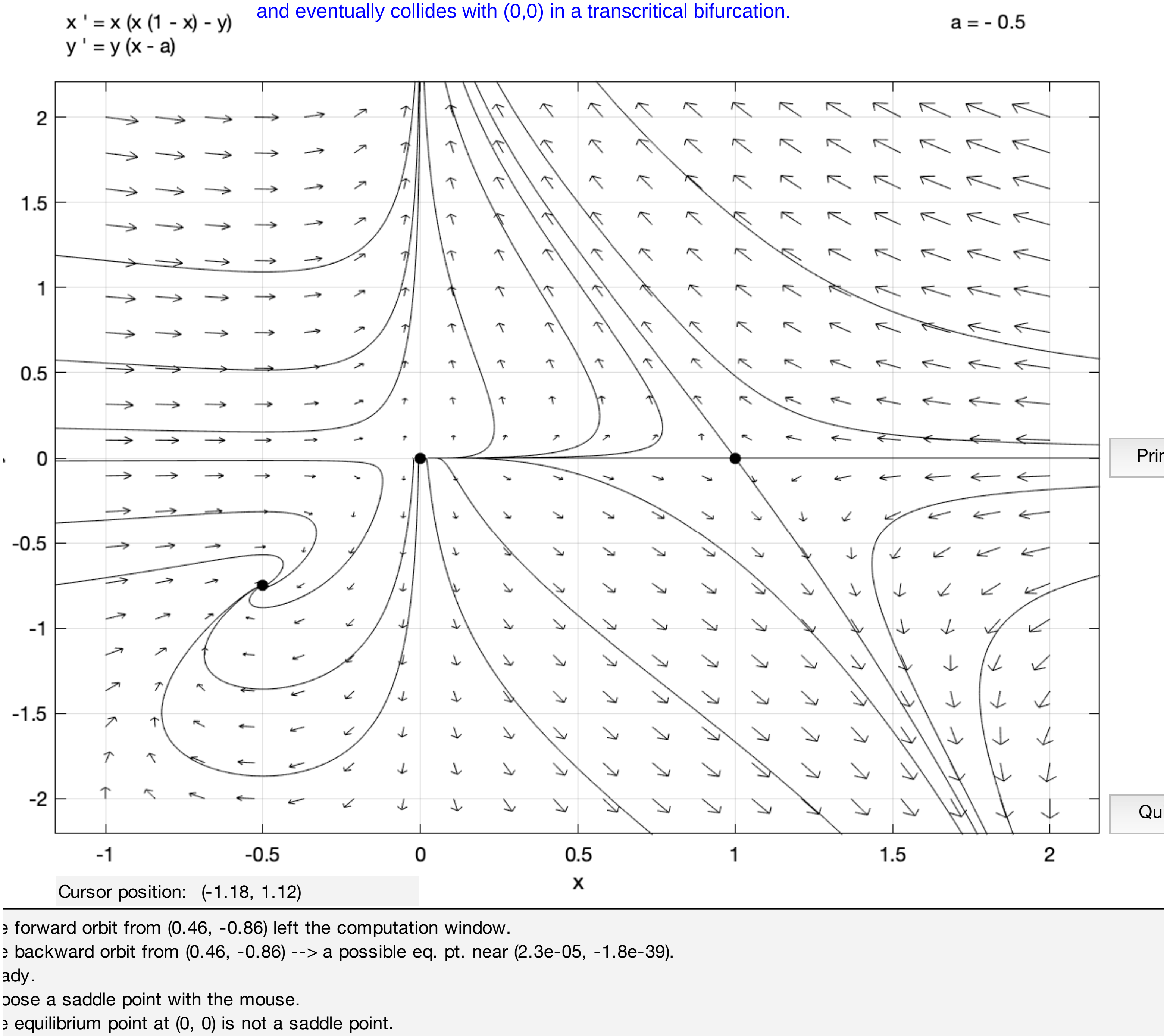
What we will see:

$a=0$		$a=0.33$	$a=0.5$	$a=1$
$x \neq 0$ $y \neq 0$ But $x<0$ Not physical	$x=0$ $y=0$ Everyone Extinct	Oscillations	$x \neq 0, y \neq 0$	$x=1$ $y=0$ Prey is extinct

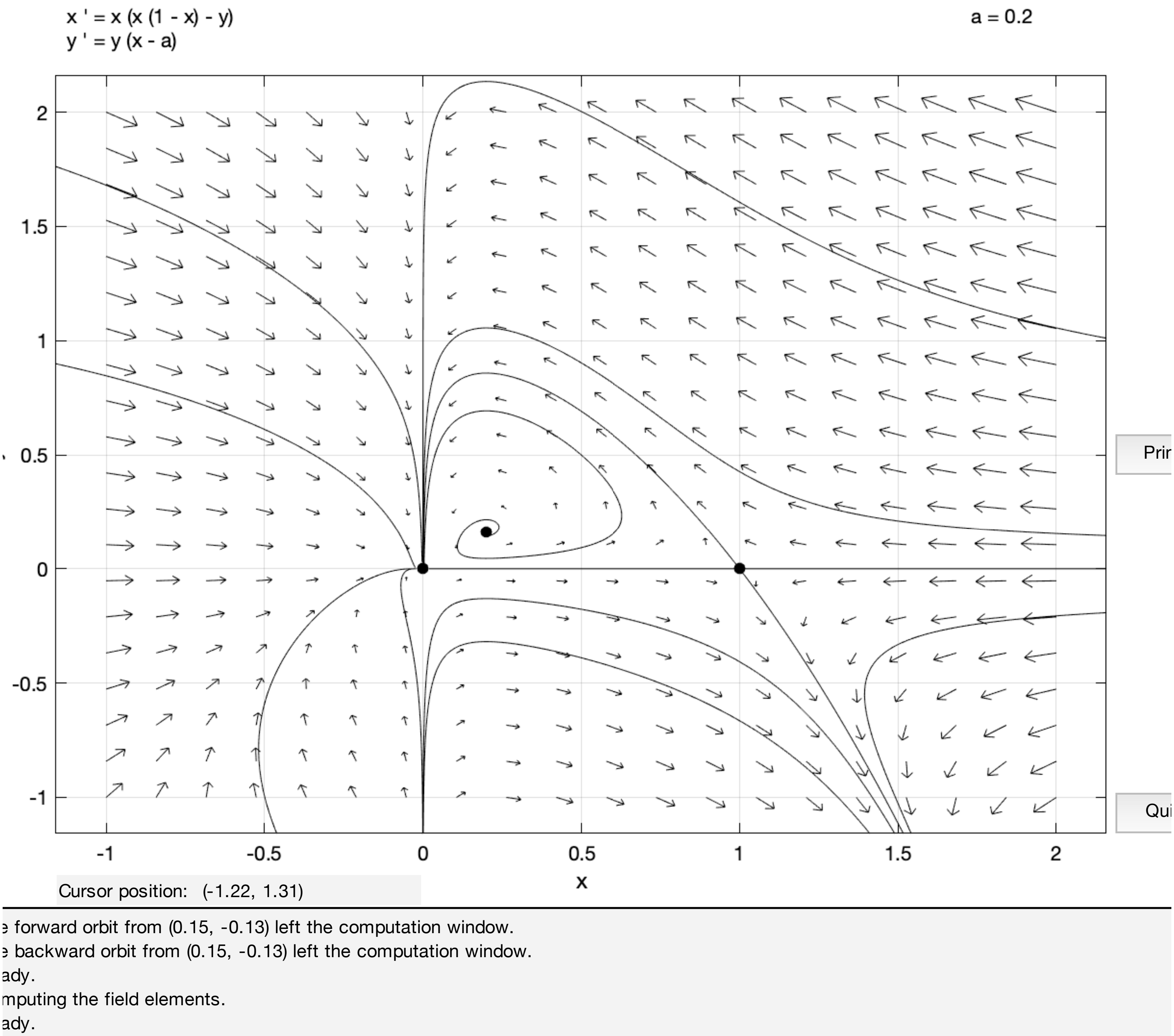
(0,0) is linearly neutral but from the phase plane we see that it is unstable. Solutions eventually move away.

(1,0) is a saddle.

(a, a(1-a)) looks to be a A.stable focus. As the parameter a is increased, this E.P. moves towards the origin and eventually collides with (0,0) in a transcritical bifurcation.

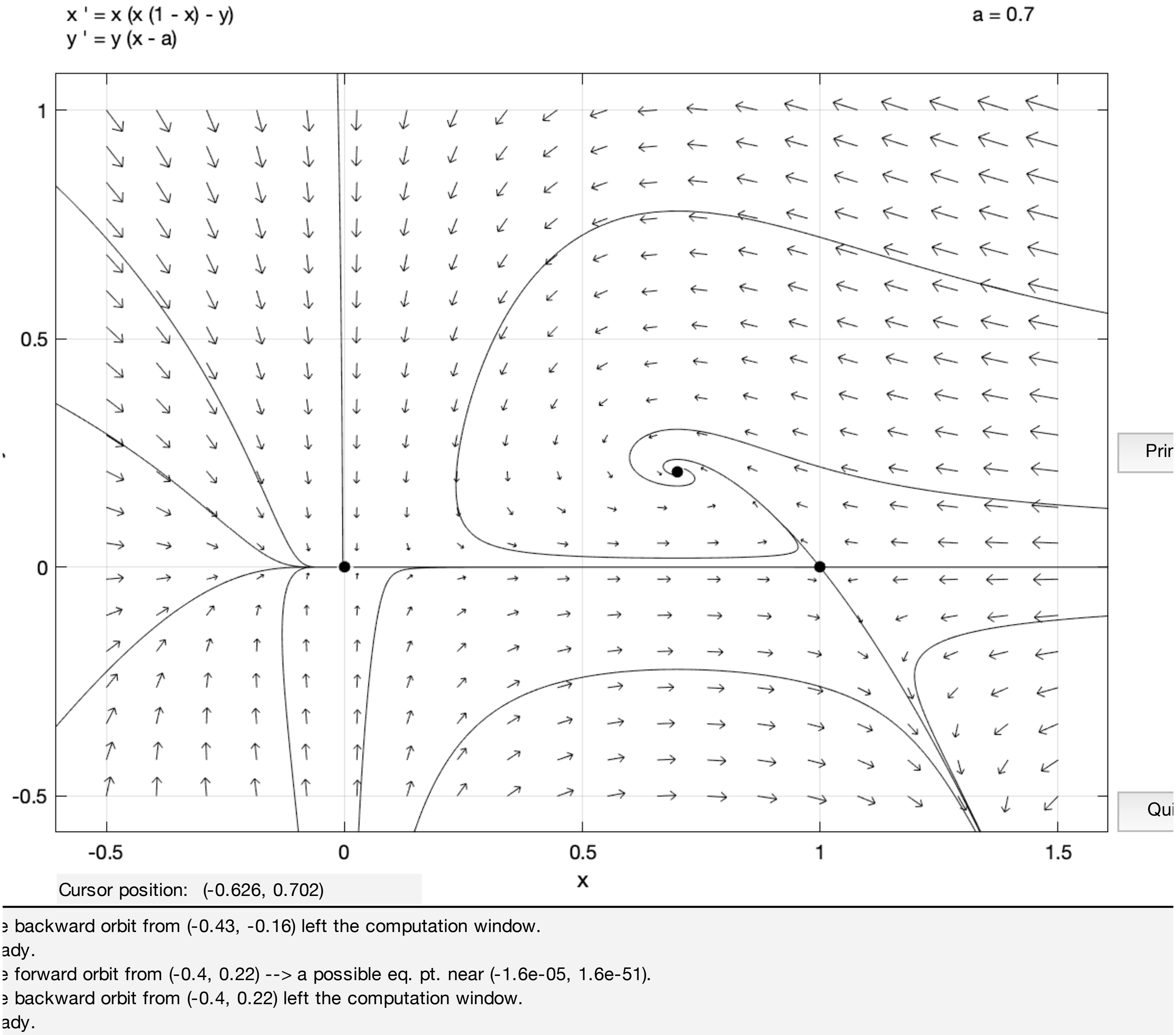


(0,0) after the bifurcation at a = 0, the origin has stable directions and unstable directions, though isn't a saddle in the usual linear sense.
(1,0) remains a saddle.
(a, a(1-a)) is now an unstable focus. Trajectories near this E.P. converge to the origin.

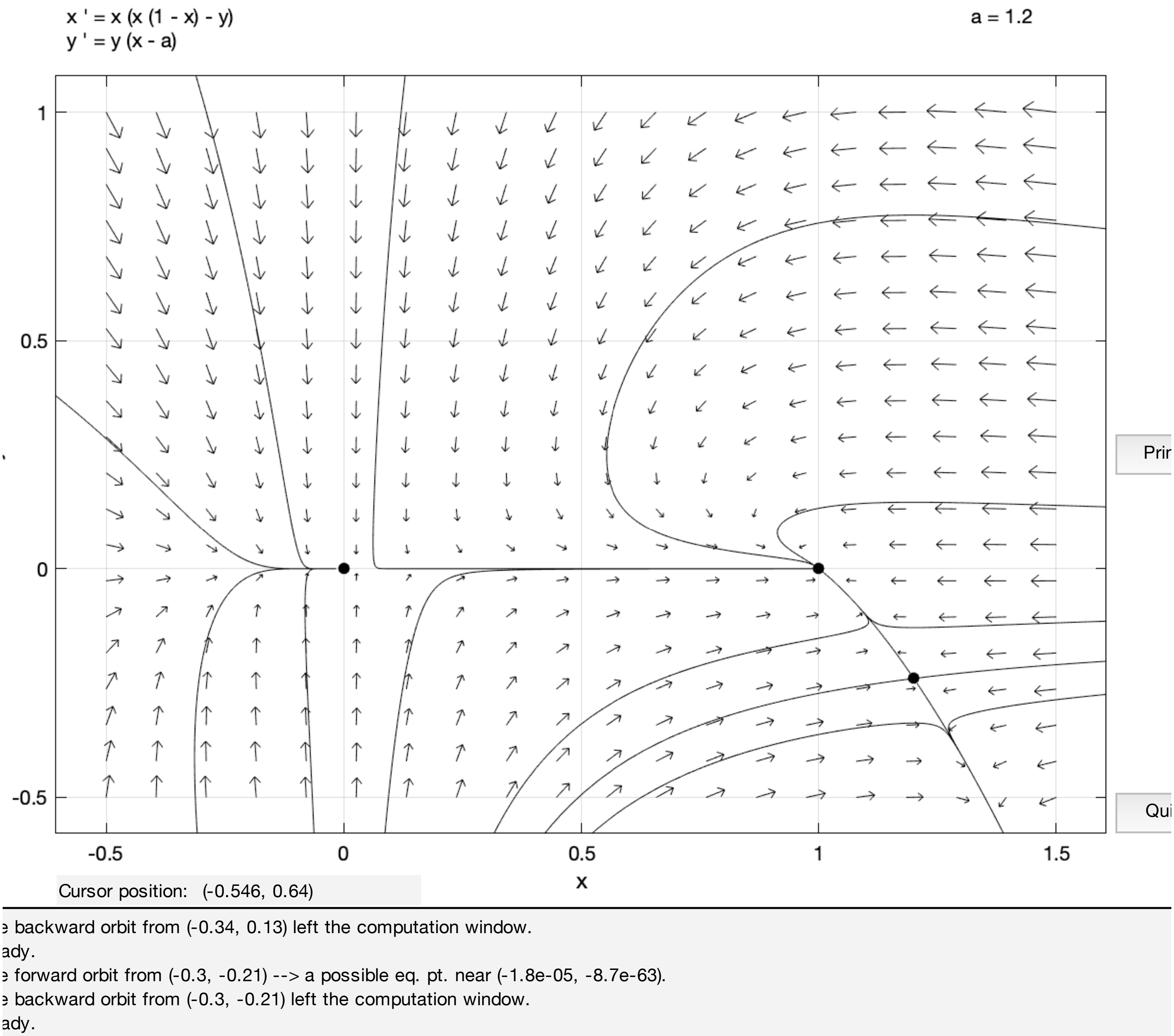


Figures out of order. Scroll down to a = 0.45

(a,a(1-1a)) is now a stable focus and the limit cycle has disappeared. As a continues to increase this E.P. moves toward the E.P. at (1,0). They collide in a transcritical bifurcation.



For a > 1, the E.P. at (1,0) that has been a saddle is now a stable node. In contrast, the E.P. at a(1-a) is now a saddle. Note that while this is all mathematically correct, from the point of view of predator-prey system this solution indicates a negative population and is not physically valid.



(0,0) continues to have stable directions and unstable directions.
(1,0) is still a saddle.
(a,a(1-a)) is an unstable spiral. However, instead of trajectories spiraling to the origin, they spiral to a stable limit cycle. The limit cycle appeared at a ~ 0.33. As a is increased, the limit cycle shrinks to collides with the E.P. point (a,a(1-a)), corresponding to a Hopf bifurcation. The fact that the limit cycle solution is stable indicates that it is a super-critical Hopf bifurcation.