Math 2339: Review problems

Instructions: Treat this assignment as an open-book, one-hour exam. If you are able to almost ace this exam, you are ready for Calculus III.

Problem 1. Basic Algebra.

(a) Reduce to simplest form the expression

$$e^{-2\ln(x)}.$$

(b) Solve for x if

$$e^{\pi} = \sqrt{2\ln(x) - 1}$$

(c) Find the unknowns x and y in the linear system

$$\begin{aligned} 2x + 3y &= 8 \\ 3x + 2y &= 7 \end{aligned}$$

(a) $e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$

(b) $(e^{\pi})^2 = 2\ln x - 1$

$\frac{e^{2\pi} + 1}{2} = \ln x$

$x = e^{\frac{1}{2}(e^{2\pi} + 1)}$

(c) $2x + 3y = 8$

$3x + 2y = 7$

$x = (8 - 3y)/2$

$\frac{3}{2}(8 - 3y) + 2y = 7$

$24 - 9y + 4y = 14$

$-5y = -10$

$y = 2$

$x = \frac{8 - 3 \cdot 2}{2} = 1$

Problem 2. Differentiation.

(a) Using appropriate rules, compute the derivative $\frac{dy}{dx}$ if

$$y = e^x \frac{x}{x^2+1}$$

(b) Use implicit differentiation to find $\frac{dy}{dx}$ if

$$x = sin\,(2xy)\,.$$

(a)

$$\frac{dy}{dx} = \frac{d}{dx}(e^x)\,\frac{x}{x^2+1} + e^x \frac{d}{dx}\left(\frac{x}{x^2+1}\right)$$

$$= e^x \frac{x}{x^2+1} + \frac{1\cdot(x^2+1) - x(2x)}{(x^2+1)^2}\, e^x$$

$$= e^x \left[\frac{x(x^2+1) + x^2 + 1 - 2x^2}{(x^2+1)^2} \right]$$

$$= e^x \left(\frac{x^3 - x^2 + x + 1}{(x^2+1)^2} \right)$$

(b)

$$x = \sin(2xy)$$

$$1 = \cos(2xy)(2)\left(1\cdot y + x\frac{dy}{dx}\right)$$

$$\frac{1}{2\cos(2xy)} = y + x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2\cos(2xy)} - y}{x}$$

$$= \frac{1}{2x\cos(2xy)} - \frac{y}{x}$$

Problem 3. Some basic integrals. Compute the following indefinite integrals:

(a) $\int 2t\sqrt{1+t^2}dt$

(b) $\int ze^z dz$

(a) Let $u = 1 + t^2$

$\frac{du}{dt} = 2t$

$dt = \frac{1}{2t} du$

$\int 2t\, u^{1/2} \frac{du}{2t} = \int u^{1/2} du$

$= \frac{2}{3} u^{3/2} + C$

$= \frac{2}{3} \sqrt{(1+t^2)^3} + C$

(b) Integrate by parts.

$\int z e^z dz$

$u = z \quad dv = e^z dz$

$\frac{du}{dz} = 1 \quad v = e^z$

$z e^z - \int e^z dz$

$z e^z - e^z + C$

Problem 4. Parametric Curves. A particle traces out the trajectory given by the parametric curve

$$x = t^3 - t$$
$$y = \sqrt{3}\,(t^2 - 1)$$

(a) Sketch the graphs of the functions $x\,(t)$ and $y\,(t)$.

(b) Using the results from part (a), sketch the trajectory of the particle in the (x, y) plane.

(c) What is the straight line distance between the points on the curve determined by $t = 0$ to $t = 2$?

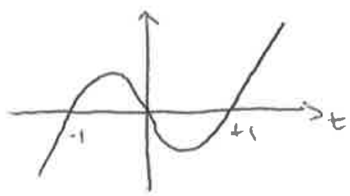
a) $x = t^3 - t$

$= t(t+1)(t-1)$

Cubic.

Roots at $0, \pm 1$

$\lim_{t \to \infty} x = +\infty$

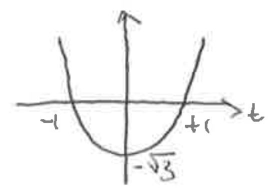


(b) $y = \sqrt{3}\,(t+1)(t-1)$

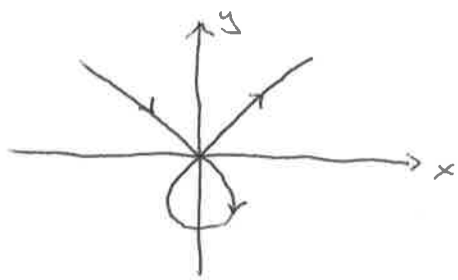
Quadratic.

Opens upward.

Roots at ± 1



(b)



(c) $t = 0$ $x = 0$ $y = -\sqrt{3}$ $(0, -\sqrt{3})$

$t = 2$ $x = 6$ $y = \sqrt{3} \cdot 3$ $(6, 3\sqrt{3})$

$$D = \left[(6-0)^2 + (3\sqrt{3} + \sqrt{3})^2\right]^{1/2}$$
$$= \left[36 + 16 \cdot 3\right]^{1/2} = \sqrt{84}$$

Problem 5. Avoiding Common Mistakes. Consider each of the following calculations performed by students. Identify where the student went wrong in each case, and identify what the student needed to have done instead. Then calculate the correct anwer.

(a) $\int \frac{1}{x^2(x+1)}\,dx = -\frac{1}{x} + \ln(x+1) + C$

(b) $\int x\cos(x)\,dx = \frac{1}{2}x^2\sin(x) + C$

(c) $\int (a+x)\,dx = \frac{1}{2}a^2 + \frac{1}{2}x^2 + C$

(a) $\frac{1}{x^2(x+1)} \neq \frac{1}{x^2} + \frac{1}{x+1}$

Correct: Use Partial Fractions $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

Solve for A, B, C

$= -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$

Integrate

$\Rightarrow \; -\ln x - \frac{1}{x} - \ln(x+1) + c$

(b) Can't integrate each term in product separately

$\int x\cos x\,dx \neq \int x\,dx \int \cos x\,dx$

Correct: Integrate by parts.

$\int x\cos x\,dx$

$\begin{matrix} 1 & \sin x \end{matrix} \quad = x\sin x - \int \sin x\,dx = x\sin x + \cos x + c$

(c) a is a constant.

Correct

$\int (a+x)\,dx = ax + \frac{1}{2}x^2 + c$