Exponential Growth & Decay

... or Death & Birth, or Deposit & Withdrawal.

There is a branch of mathematical biology called “Population Dynamics,” where the competition between species is studied. This is important in environmental resource management. “Epidemiology” is very similar in that there is a competition between those who are susceptible, those who are infected and those who are recovered, from a particular disease.

Let $P$ be the number of items of interest, for example, “population.”

\[
\frac{dP}{dt} = \text{Rate of change} \text{ of the population.}
\]

\[
\frac{dP}{dt} \frac{1}{P} = \frac{\text{Rate of change}}{\text{population}} = \frac{1}{P} \frac{dP}{dt}
\]

\[
= \text{Growth Rate}
\]

Growth rate measures the Rate of change with respect to the population size, i.e., the relative rate of change. The distinction is important.

- Suppose $\frac{dP}{dt} = k$.
  - No matter what the size of the population, there are always $k$ new elements in a given time.
  - Solve by integration.
  - $P(t) = kt + C$, apply IC $C = P(0)$, so $P(t) = P(0) + kt$.
  - Results in linear growth.
  - Models number of people in Bush Stadium. No matter how many have been admitted, the turnstile gates allow only a fixed number to enter over a given time.

- Suppose $\frac{1}{P} \frac{dP}{dt} = k$ or $\frac{dP}{dt} = kP$.
  - The rate of change depends on the population size. For example, this reflects the fact that with more members in a population there can be more births.
  - Solve by separation.
  - $P(t) = Ce^{kt}$, apply IC $C = P(0)$, so $P(t) = P(0)e^{kt}$.
  - Results in exponential growth.

- Which model is more accurate depends on the application.

Decay is just the negative of growth.

- $\frac{dP}{dt} = -k$ so $P(t) = P(0) - kt$.
  - Linear decrease.
  - People leave Bush Stadium at a fixed rate.

- $\frac{dP}{dt} = -kP$ so $P(t) = P(0)e^{-kt}$.
  - Exponential decay.
  - The more members of a population, the more deaths there are.