Math 2343: Final Exam, Fall 2004

Instructions:

- No notes, books or calculators.
- Do not write on the exam page. Only work in the blue book will be graded.
- Clearly label your answers with the corresponding problem number.

1. [10 pts]
Solve the following first-order differential equations:

(a) \( \frac{dy}{dx} = yx^2 \sin(x^3 + \pi) \quad y(0) = 1 \),

(b) \( 2\frac{dy}{dx} + 4y = 2e^8x, \quad y(1) = 0 \)

2. [10 pts]
A colony of fire ants reproduces according to an exponential growth law with growth rate \( k_1 = 2.0 \text{ day}^{-1} \). Their natural death rate is \( k_2 = 1.5 \text{ day}^{-1} \). The initial size of the population is \( 10^7 \) ants.

(a) Formulate the differential equation and initial condition for the number of ants.

(b) Solve the differential equation and determine how many days it takes for the colony population to double.

(c) If insecticide is applied to the colony, then \( l = 10^6 \text{ ants/day} \) are killed. Formulate and solve the differential equation under these new conditions.

(d) Is this enough insecticide to kill the ant colony?

3. [10 pts]
The SMU outdoor pool holds \( 2.5 \times 10^6 \text{ l} \) of chlorinated \( \text{H}_2\text{O} \) with chlorine concentration of \( 10^{-4} \text{ g/l} \). Due to heavy rains and run-off pure \( \text{H}_2\text{O} \) is entering the pool at a rate of \( 10^4 \text{ l/h} \) (about 3 linear inches in 10 hours). To prevent overflow the drain has been opened and pool water flows out at \( 0.9 \times 10^4 \text{ l/h} \). Evaporation is negligible. Find the ODE and ICS for the amount of chlorine in the pool as a function of time BUT DO NOT SOLVE.

4. [10 pts]
Given

\[ \frac{dy}{dt} = f(y) = r - y^2, \]

for BOTH \( r = +1 \) and \( r = -1 \) ...

(a) Sketch \( f(y) \) and use it to sketch the phase line. Label the equilibrium points and indicate their stability.

(b) Make a neat and accurate of the direction field.

(*) Points may be deducted for ambiguous sketches.

For problems (5)-(9), use the differential equation below:

\[ y'' + 2y' + 5y = f(t), \]

5. [5 pts] Rewrite the ODE as a system of two first-order ODEs.
(6) [5 pts] Find the homogeneous solution.

(7) [10 pts] Find the particular solution using \( M \) of \( UC \), then use the ICs to fully determine the complete solution solving for all unknown coefficients, if

\[
f(t) = 2e^{-t}, \quad y(0) = \frac{1}{3}, \quad y'(0) = -\frac{1}{3}.
\]

(8) [10 pts] Give the form of the particular solution but DO NOT SOLVE FOR THE COEFFICIENTS if

\[
f(t) = t^2e^{-t}\sin(2t) + (2t^3 + 3t^2)e^{-t}
\]

(9) [10 pts] Find the complete solution using LAPLACE TRANSFORMS if

\[
f(t) = 2e^t \quad \text{and} \quad y(0) = 1, \quad y'(0) = -2.
\]

(10) [10 pts]
For the system of ODEs:

\[
\begin{align*}
x' &= 1 - x \\
y' &= y - 1
\end{align*}
\]

(a) Determine and sketch the equilibrium point and nullclines in the phase plane.
(b) Solve the phase plane equation and plot the integral curves.
(c) State the stability and type for each equilibrium point.

(11) [10 pts]
A linear mass-spring system is described by the differential equation

\[
my'' + by' + ky = f(t), \quad y(0) = 1, \quad y'(0) = 1.
\]

(a) For an unforced frictionless mass-spring with a mass of 1 and a spring constant of 4, determine the resulting motion. (i) Express your answer in amplitude-phase form. (ii) Sketch the solution.
(b) The same frictionless mass-spring is used in a mechanical system subject to harmonic forcing of the form

\[
f(t) = F \cos \omega t.
\]

What frequency \( \omega \) will lead to resonance? Describe resonance with words, sketches, plots, equations, etc..
(c) A damper (resistance/friction) is added to the system to limit the effects of resonance. What is the maximum level of damping so that the system is under-damped?
Possibly useful formulas:

\[
\begin{align*}
\cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\
\sin(a \pm b) &= \sin a \cos b \pm \sin b \cos a
\end{align*}
\]

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