Math 2343: Exam 1

Instructions:

- No notes, books or calculators.
- Do not write on the exam page.
- Clearly label your answers with the corresponding problem number.

(1) [(a) = 5pts, b-f = 10 pts each = 55 pts]
Solve the following ODEs:

(a-integrate) \[ \frac{dy}{dx} = -5x^2e^{2x^3}, \ y(0) = 1, \]

(b-separate) \[ \frac{dy}{dx} = y^2\sin 2x, \]

(c-separate) \[ x\frac{dy}{dx} = x\sin(y^{2/3})e^{3x}, \ y(x_0) = y_0, \]

(d-IF) \[ \frac{dy}{dx} + 2ky = l, \ k \text{ and } l \text{ are constants.} \]

(e-IF) \[ \frac{dy}{dx} + 2xy = 2x, \]

(f-IF) \[ (x^2 + 1)\frac{dy}{dx} + xy = 5x^3 + 5x, \ y(0) = \frac{5}{3}, \]

(2) [10 pts]
A tree fungus that reproduces according to an exponential growth law is affecting a forest such that the number of infected acres doubles in 30 days. The Forest Service is treating the fungus by spraying “tree deodorant” from planes and can eradicate the fungus by 100 acres per 30 days. However, the Forest Service did not realize there was a problem until 10000 acres were already infected.

(a) First determine the growth rate k.

(b) Then formulate and solve the DE taking into account both the growth of the fungus and the interference by the Forest Service. If unsuccessful in (a) use \( k = \ln 2/20 \).

(c) Can the forest be saved from the fungus such that it is totally eradicated?

(3) [10 pts]
You are in charge of a party and plan to have “Orange Spritzes” for a beverage that ideally contains 70% orange juice and 30% soda water. Thus, your initial batch of 10 L has a juice concentration of 0.75 (L per L implies no units). Thirsty party goers are drinking 3 L per hour of the Spritz. You delegate a friend to make sure the punch bowl stays supplied. Unfortunately, they are not too bright and only add soda water at a rate of 2 L per hour. Also, it is very hot out such that water (just water) evaporates at 0.25 L per hour. Formulate BUT DO NOT SOLVE an ODE and IC for the amount of orange juice in the Spritz as a function of time.
(4) [10 pts]
For the ODE
\[ \frac{dy}{dx} = \frac{y}{x} \]
(a) Sketch the direction field for \(-2 \leq x \leq 2\) and \(-2 \leq y \leq 2\).
(b) Does the solution exist for all \(x\) and \(y\)? Is the solution unique for all \(x\) and \(y\)? If not, where not and why not.
(c) Solve for the solution \(y\) by separation.

(5) [15 pts]
For the autonomous ODE
\[ y' = f(y) = y(y - 2)(y + 1) \]
(a) Determine the equilibrium points.
(b) Examine \(f(y)\) to sketch the phase line and indicate the stability of the equilibrium points.
(c) Sketch the direction field for \(-1 \leq x \leq 1\) and \(-2 \leq y \leq 3\). Plot the solution curves corresponding to the initial condition \(y(0) = -1/2\) and \(y(0) = 1/2\).
(e) Now consider \(y' = g(y)\) where \(g(y) = -f(y)\). What are the equilibrium points and their stability.

A Historical Mixture Problem: John Joly (1899) devised a new geologic technique for determining the Earth’s age. He maintained that all the salt in the oceans had come from mineral deposits that had eroded and dissolved. He further assumed (erroneously) that the amount of salt in the ocean could not decline. By multiplying the salinity of the ocean by its volume and dividing by the annual increase, he determined that the brackish sea had developed over 80-90 million years.

A Historical Decay Problem: Arthur Holmes (1926) was the primary author of a report for the National Academy of Sciences in which the committee agreed unanimously that radioactive dating was the only reliable geologic timescale. By this time, the constants of radioactivity were firmly established and other sources of problems such as specimen selection and lead isotopes were understood. The scale has been further refined over the last 70 years. Currently, the record for the oldest known rocks on Earth is 3.96 billion years.