A VARIABLE ORDER WAVELET METHOD FOR THE SPARSE REPRESENTATION OF LAYER POTENTIALS IN THE NON-STANDARD FORM.

JOHANNES TAUSCH

Abstract. We discuss a variable order wavelet method for boundary integral formulations of elliptic boundary value problems. The wavelet basis functions are transformations of standard nodal basis functions and have a variable number of vanishing moments. For integral equations of the second kind we will show that the non-standard form can be compressed to contain only $O(N)$ non-vanishing entries while retaining the asymptotic converge of the full Galerkin scheme, where $N$ is the number of degrees of freedom in the discretization.

1. Introduction

Elliptic boundary value problems with constant coefficients can often be cast into integral equations of the second kind on the boundary surface. The resulting equations usually involve layer potentials of the form

$$ (Ku)(x) := \int_{S} k(x, y)u(y) dS_y, $$

If $K$ is a single layer potential, the kernel $k(x, y)$ is given by

$$ k(x, y) = G(|x - y|) $$

and in case of a double layer potential the kernel is of the form

$$ k(x, y) = \frac{\partial}{\partial n_y} G(|x - y|). $$

Here, $G$ is a Green’s function of the differential operator.

A common discretization scheme for layer potentials is the Galerkin method using piecewise polynomial elements. If \{\chi_i : i = 1, \ldots, N\} is a basis of the finite element space, then the Galerkin discretization of $K$ is a matrix with coefficients

$$ K_{i,j} = \langle \chi_i, K\chi_j \rangle. $$

We will refer to such a coefficient as the interaction of function $\chi_i$ with $\chi_j$. A natural choice of a basis is a nodal basis, i.e., the box-functions for piecewise constant and hat-functions for piecewise linear elements. Since $K$ is an integral operator the matrix in (4) is densely populated and therefore storage and complexity of linear algebra is $O(N^2)$. A remedy to this problem is to perform matrix vector products approximately to reduce the complexity to almost linear cost. Examples of such algorithms are the Fast Multipole Method [8, 9], panel clustering [10, 13], or FFT-based techniques, e.g. [12, 2].

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A different solution of the difficulties associated with dense matrices is to apply wavelets for the discretization instead of the usual nodal basis functions. In addition to compression of integral operators, wavelet methods provide a framework for preconditioning and adaptive mesh generation [5, 4, 3]. The first step in this direction was the paper [1], where wavelets have been applied to find approximate representations of integral operators on an interval which contain $O(N \log \epsilon)$ non-zero entries. Here, $\epsilon$ is a parameter for the accuracy. These results have motivated a number of investigations to apply wavelet discretizations for surface integral operators.

The main thrust of research in the recent years has been to construct wavelets in a parameter space of the boundary manifold and to apply a compression scheme to the standard form of the integral operator. The papers [22, 23] show that this approach leads to nearly optimal compression schemes for integral equations of the second kind. An implementation is reported in [11]. Moreover, the monograph [15] extends this methodology to derive optimal compression algorithms for operator equations of non-zero order.

In an earlier paper we have described an alternative approach to compress the non-standard form using wavelet-like functions which are directly constructed on the surface instead of a parameter space [20, 21]. The 'wavelets' of this basis are linear combinations of standard nodal basisfunctions and are orthogonal to polynomials in three variables. This approach leads to sparse representations of integral operators in complicated geometries, unstructured grids and comparatively coarse discretizations.

The present paper is concerned with a much refined error analysis of this approach. We consider a boundary integral equation of the second kind which has been discretized with piecewise constant elements. We will show that if the number of vanishing moments is suitably adjusted to the scale of the wavelet, the non-standard form can be compressed to contain $O(N)$ non-vanishing entries, while preserving the asymptotic convergence of the uncompressed Galerkin scheme.

So far, optimal results have been known only for compression with the standard form; see, Schneider [15]. We pursue the non-standard form, because it involves only interactions of basisfunctions at the same level. Therefore, algorithms for setting up the matrix and applying the matrix in wavelet domain are much simpler to realize.

We will describe a fully discrete method. That is, we discuss a quadrature scheme for computing the entries of the non-standard form. These are singular or nearly singular double surface integrals for which quadrature schemes must be designed carefully so that the optimality of the method is preserved. For wavelets defined in a parameter space, the integration is usually done in the parameter space as well. Different strategies based on this approach have been described, for instance in [22, 7]. Since there are no parameterizations in our setting, we cannot resort to such techniques. Our algorithm is in essence the fast wavelet transform, or pyramid scheme, applied to a matrix instead of the usual vector. The complexity for setting up the compressed non-standard form is $O(N)$ where the accuracy of quadratures is chosen not to affect the convergence.

We conclude with an algorithm for setting up the basis in $O(N)$ operations and with a numerical example. For more extensive numerical experiments we refer to [18].
2. Wavelet Basis

For the asymptotic estimates that follow, we assume that the boundary surface $S$ is smooth and divided into an initial coarse triangulation which is several times uniformly refined. Thus there is a sequence of triangulations $\mathcal{T}_h$, where $h$ is a parameter for the meshwidth. We note however, that the wavelet compression scheme can also be applied effectively for much more general, unstructured meshes.

The optimal compression result of Section 3 applies for integral equations of the second kind discretized with piecewise constant elements. However, our wavelet basis can also be constructed for elements of an arbitrary order, which is the case we will consider in this section.

The finite element space $X_h$ consists of functions which are piecewise polynomial on $\mathcal{T}_h$. The obvious choice of a basis $\{\chi_1, \ldots, \chi_N\}$ of $X_h$ is a nodal basis. That is, there are node points $v_1, \ldots, v_N$ on the surface such that

$$\chi_i(v_j) = \delta_{i,j}.$$ 

If $X_h$ is the space of the piecewise constants, the nodes are the centers of the triangles and the nodal basis functions are the box-functions. In the piecewise linear case, the nodes are the vertices and the nodal basis functions are the hat-functions.

The surface $S$ is embedded in a top level cube. The cube is subdivided into eight cubes of equal size which are the children or level-one cubes. The subdivisions are continued until the the finest-level cubes contain at most a predetermined number of panels. Thus the finest level $L$ is linked to $h$ in that $L$ is incremented with each mesh refinement. The set of cubes that have a non-empty intersection with $S$ at the $L$-th refinement level is denoted by $C_L$.

Further Notations.

1. A cube is empty if it contains no node points. The non-empty children of cube $\nu$ are denoted by $\mathcal{K}(\nu)$.
2. The subspace of $X_h$ that is spanned by nodal basis functions whose nodes are in $\nu$ is denoted by $X_\nu$. That is,

$$X_\nu := \{ f_h \in X_h : f(v_i) = 0, \text{ if } v_i \text{ is a node outside } \nu \}.$$ 

The index set of nodal functions $\chi_i$ that span $X_\nu$ is denoted by $N_\nu$.
3. $S_\nu \subset \mathcal{T}_h$ is the maximal support of a function in $X_\nu$.
4. $B_\nu$ is the smallest axisparallel box in $\mathbb{R}^3$ that encloses $S_\nu$, $x_\nu$ is $B_\nu$’s center and $\rho_\nu$ is the half-length of its diagonal.
5. The $a$-th moment of a function $f_\nu \in X_\nu$ is given by

$$m_\nu^a(f_\nu) := \int_{S_\nu} h_{a,\nu}(x)f_\nu(x) dS_x,$$

where $h_{a,\nu}$ is either a three-variate monomial $h_{a,\nu}(x) = (x - x_\nu)^a$, $x \in B_\nu$ or some approximation of the monomial. We will consider approximate monomials in Section 6. Furthermore, we set

$$\Pi_{p,\nu} := \text{span}_{a} h_{a,\nu}.$$ 

To understand some of the more technical issues in this article it is important to realize that the cubes in a given level coincide with an orthogonal lattice, whereas the enclosing boxes can have different sizes, be off-centered and may protrude their cubes to some extent. We assume that the maximal protrusion $\delta$ is no bigger than a quarter of the sidelength of the cube, see Figure 1.
Central to the construction of the wavelet basis are moment matrices. Suppose \( \phi := \{ \phi_1, \ldots, \phi_n \} \) is a set of functions in \( X_\nu \). The moment matrix of order \( p \) is the matrix
\[
M(\phi) = [m^\alpha(\phi_j)]_{|\alpha| \leq p, 1 \leq j \leq n}
\]
If the set \( \phi \) is transformed into another set \( \tilde{\phi} \)
\[
\tilde{\phi}_j = \sum_{i=1}^{n} q_{i,j} \phi_i, \quad j = 1, \ldots, n
\]
then the moment- and transformation matrices satisfy
\[
M_\nu(\tilde{\phi}) = M_\nu(\phi) Q.
\]
For the transformation in (6) we will also use the notation \( \tilde{\phi} = Q \phi \).

**Finest Level.** To define the finest level transformation, compute the singular value decomposition of the moment matrix \( M(\chi_\nu) \), where \( \chi_\nu := \{ \chi_i \}_{i \in N_\nu} \)
\[
M(\chi_\nu) = U_\nu \Sigma_\nu Q_\nu^T, \quad \nu \in C_L,
\]
and set
\[
\Phi_{\nu,j} = \sum_{i \in N_\nu} q_{i,j} \chi_i, \quad j \in I_\nu,
\]
\[
\Psi_{\nu,j} = \sum_{i \in N_\nu} q_{i,j} \chi_i, \quad j \in J_\nu,
\]
where \( q_{i,j} \) are the coefficients of \( Q_\nu \) and \( J_\nu \) and \( I_\nu \) are indices corresponding to vanishing and non-vanishing columns of \( \Sigma_\nu \). The spaces \( V_\nu \) and \( W_\nu \) are given by
\[
V_\nu = \text{span}_{j \in I_\nu} \{ \Phi_{\nu,j} \} \quad \text{and} \quad W_\nu = \text{span}_{j \in J_\nu} \{ \Psi_{\nu,j} \}
\]
By construction, the space \( W_\nu \) is the subspace of functions in \( X_\nu \) with vanishing moments up to order \( p \).

**Coarser Levels.** The basis functions in level \( l, l < L \) are transformations of \( \Phi \)-functions in level \( l + 1 \). For \( \nu \in C_l \) we denote the set of all \( \Phi \)-functions in \( \mathcal{K}(\nu) \) by \( \phi_\nu \), that is
\[
\phi_\nu = \{ \Phi_{\mu,i} : \mu \in \mathcal{K}(\nu), i \in I_\mu \} = \left\{ \Phi_i : i \in \tilde{I}_\nu \right\},
\]
where \( \tilde{I}_\nu \) is the index set of \( \Phi \)-functions in the kids of \( \nu \).
The coefficients of the transformation are given by the singular value decomposition $M(\phi_\nu) = U_\nu \Sigma_\nu Q_\nu^T$. The $\Phi$- and the $\Psi$-functions of $\nu$ are linear combinations

\begin{align}
\Phi_{\nu,j} &= \sum_{i \in I_\nu} q_{i,j} \Phi_i, \quad j \in J_\nu, \\
\Psi_{\nu,j} &= \sum_{i \in I_\nu} q_{i,j} \Psi_i, \quad j \in J_\nu.
\end{align}

Furthermore, we set $V_\nu = \text{span}\{\Phi_{\nu,j}\}$ and $W_\nu = \text{span}\{\Psi_{\nu,j}\}$. By construction, the functions in $W_\nu$ have vanishing moments.

The single- and multi scale spaces in level $l$ are defined as

$$V_l = \bigoplus_{\nu \in \mathbb{C}_l} V_\nu, \quad \text{and} \quad W_l = \bigoplus_{\nu \in \mathbb{C}_l} W_\nu.$$ 

The single scale spaces are nested subspaces of $X_h$. The multiscale spaces consist of the differences of the single-scale spaces of two levels. Thus we have the decomposition of the finite element space

$$X_h = W_L \oplus W_{L-1} \oplus \cdots \oplus W_2 \oplus V_2.$$ 

**Multiresolution Analysis.** A function given by its coefficients in the the standard basis can be transformed efficiently into the wavelet basis with the well known pyramid scheme. We will describe it briefly.

If $f \in V_\nu \oplus W_\nu$ then $f$ can be written as a linear combination of the $\Phi$-functions of the children’s cubes, as well as a linear combination of the $\Phi$- and $\Psi$-functions of cube $\nu$

$$f = \sum_{i \in I_\nu} f_{i,\nu,i} \Phi_i = \sum_{j \in J_\nu} \hat{f}_{\nu,j} \Phi_{\nu,j} + \sum_{j \in J_\nu} \hat{f}_{\nu,j} \Psi_{\nu,j}.$$ 

From (7), (8), (9) and (10) and the orthogonality of $Q_\nu$ it follows that

$$\hat{f}_{\nu,j} = \sum_{i \in I_\nu} q_{i,j} f_{i,\nu,i}, \quad j \in J_\nu \quad \text{and} \quad \hat{f}_{\nu,j} = \sum_{i \in I_\nu} q_{i,j} f_{i,\nu,i}, \quad j \in J_\nu.$$ 

Thus the coefficients in a coarse level are orthogonal transformations of the coefficients in the next finer level. We set

$$\hat{f}_1 = \sum_{i \in I_\nu} \hat{f}_{i,\nu,i} \Phi_{i,\nu,i} \in V_1 \quad \text{and} \quad \hat{f}_1 = \sum_{i \in I_\nu} \hat{f}_{i,\nu,i} \Psi_{i,\nu,i} \in W_1.$$ 

The transform starts in the finest level and calculates the coefficients at increasingly coarser levels

$$f \rightarrow \hat{f}_L \rightarrow \hat{f}_{L-1} \rightarrow \cdots \rightarrow \hat{f}_2$$

For the inverse transform, it follows from (13) that

$$f_{\nu,i} = \sum_{j \in I_\nu} q_{i,j} \hat{f}_{\nu,j} + \sum_{j \in J_\nu} q_{i,j} \hat{f}_{\nu,j}$$
and thus the inverse transform is calculated starting at the coarsest scale

$$
\tilde{f}_2 \rightarrow \tilde{f}_3 \rightarrow \ldots \rightarrow \tilde{f}_L \rightarrow f.
$$

(16)

The function $f \in X_h$ has the two representations

$$
f(x) = \sum_{i=1}^{N} f_i \chi_i = \sum_{\nu \in \tilde{C}_h, i \in \tilde{J}_\nu} \tilde{f}_{\nu,i} \Phi_{\nu,i} + \sum_{\nu \in \tilde{C}_h} \tilde{f}_{\nu,i} \Phi_{\nu,i}
$$

Suppose now that the nodal basis functions are normalized such that $||\chi_i||_0 = 1$, then it is well known that the $L_2$-norm of the coefficients of the nodal basis are equivalent to the $L_2(S)$-norm of $f$, with constants independent of $h$. Furthermore, the wavelet transform is orthogonal in the coefficients which implies the following norm equivalence

$$
||f||^2_0 \sim \sum_{i=1}^{N} f_i^2 = \sum_{\nu \in \tilde{C}_h, i \in \tilde{J}_\nu} \tilde{f}_{\nu,i}^2 + \sum_{\nu \in \tilde{C}_h} \tilde{f}_{\nu,i}^2
$$

(17)

In particular, for $\tilde{f}_i \in W_i$,

$$
||\tilde{f}_i||^2_0 \sim \sum_{\nu \in \tilde{C}_h, i \in \tilde{J}_\nu} \tilde{f}_{\nu,i}^2,
$$

(18)

where $\tilde{f}_1 := \tilde{f}_2$. Combining (17) and (18) shows the stability of the multiscale transform in the $L^2(S)$-norm

$$
||f_h||^2_0 \sim \sum_{i=1}^{L} ||\tilde{f}_i||^2_0,
$$

(19)

and moreover,

$$
||\tilde{f}_i||_0 \leq C ||f_h||_0.
$$

(20)

3. Compression of integral operators

Integral operators can be efficiently compressed in wavelet basis. There are two ways to do this, namely via the standard and the non-standard form but we limit the discussion to the latter form. Following [1], we consider the decomposition of the bilinear form

$$
\langle f_h, K g_h \rangle = \sum_{\nu, \nu' \in \tilde{C}_h} \langle \tilde{f}_\nu, K \tilde{g}_{\nu'} \rangle + \sum_{\nu \in \tilde{C}_h} \sum_{\nu' \in \tilde{C}_h} \langle \tilde{f}_\nu, K \tilde{g}_{\nu'} \rangle + \langle \tilde{f}_\nu, K \tilde{g}_{\nu'} \rangle + \langle \tilde{f}_\nu, K \tilde{g}_{\nu'} \rangle.
$$

(21)

The key idea of wavelet compression is to truncate terms from (21) that correspond to cubes which are sufficiently separated. For the cube $\nu$ the set of neighboring cubes that are retained in the non-standard form is denoted by $N(\nu)$. The truncated
bilinear form is

$$\langle f_h, K^c g_h \rangle = \sum_{\nu, \nu' \in C_2} \langle \hat{f}_\nu, K \hat{g}_{\nu'} \rangle + \sum_{l=2}^{L} \sum_{\nu' \in N^l(\nu)} \langle \hat{f}_\nu, K \hat{g}_{\nu'} \rangle + \langle \hat{f}_\nu, K \hat{g}_{\nu'} \rangle + \langle \hat{f}_\nu, K \hat{g}_{\nu'} \rangle.$$  

The truncation strategy will be described below in more detail.

In order to calculate a matrix vector product of a function $g_h \in X_h$, the function must be transformed into the wavelet basis via (14). Then the (incomplete) coefficients of the potential vector $u_h$ are calculated

$$u_{\nu} = \sum_{\nu' \in N(\nu)} \langle \Phi_{\nu}, K \Phi_{\nu'} \rangle \hat{g}_{\nu'}, \quad \nu \in C_1, \; l = 2, \ldots, L\)$$

$$\hat{u}_{\nu} = \sum_{\nu' \in N(\nu)} \langle \Phi_{\nu}, K \Phi_{\nu'} \rangle \hat{g}_{\nu'} + \langle \Psi_{\nu}, K \Psi_{\nu'} \rangle \hat{g}_{\nu'}, \quad \nu \in C_1, \; l = 2, \ldots, L.$$  

The interaction $\Phi$-functions in the coarsest levels are accounted for by adding

$$\hat{u}_{\nu} := \hat{u}_{\nu} + \sum_{\nu' \in C_2} \langle \Phi_{\nu}, K \Phi_{\nu'} \rangle \hat{g}_{\nu'}, \quad \nu \in C_2.$$  

The potential vector $u_h \in X_h$ is recovered by the additive inverse transform. That is, in (16) the transforms of the coarser levels are added to the already computed $\Phi$-components of the potential vector.

**Truncation Scheme.** The separation ratio of two cubes $\nu, \nu' \in C_1$ is defined for $\nu \neq \nu'$ as

$$\eta_{\nu, \nu'} := \min_{y \in S_{\nu}, \; x \in S_{\nu'}} \max_{\nu \in S_{\nu}} \frac{|y - \bar{y}|}{|x - \bar{y}|}$$

and $\eta_{\nu, \nu} = \infty$. The neighbors of cube $\nu \in C_1$ are the set

$$N(\nu) = \{ \nu' \in C_1 : \max (\eta_{\nu, \nu'}, \eta_{\nu', \nu}) > \eta \}$$

where $\eta < 1$ is a parameter at our disposal.

**Assumption 3.1.** Let $x \in S_{\nu}, \; y \in S_{\nu'}, \; \nu' \notin N(\nu)$, and let $T_p$ be the $p$-th order generalized Taylor polynomial of $y \rightarrow G(|x - y|)$ with center $\bar{y} \in S_{\nu}$, i.e.,

$$T_p(y) = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} D^\alpha G(x - \bar{y}) h_{\alpha, \nu}(y).$$

Then there are constants $c$ and $\kappa$ such that for $\lambda \in \{0, 1\}$

$$\left| \frac{\partial^\lambda}{\partial y_0^\lambda} \left( G(|x - y|) - T_p(y) \right) \right| \leq \frac{c p^\lambda}{|x - \bar{y}|} \left( \frac{|y - \bar{y}|}{|x - \bar{y}|} \right)^p$$

It can be shown that, if $h_{\alpha, \nu}$ are the monomials, the above assumption is satisfied for the Green’s functions that arise for the Laplace and Helmholtz equation, linear elasticity and Stokes flow. See, e.g., [16].

In (29) the expansion order $p$ controls the error of the maximum norm. For the error of the Galerkin discretization the $L^2$-norm is of greater importance, where the size of the support of a wavelet is part of the error estimate. Therefore it suffices to use wavelets with fewer vanishing moments in the fine levels than in the coarse
levels. We will show in the following that an optimal scheme can be obtained if the order in level \( l \) is given by

\[
p_l = p_L + L - l.
\]

The order in the finest level \( p_L \) is fixed as the mesh is refined and typically very small.

**Complexity estimates.** For each refinement of a given discretization the number of levels in the cube hierarchy is incremented. If a non-empty cube is refined, there are on average four non-empty children. Furthermore, the number of non-empty cubes in the finest level is proportional to the number of panels. Therefore, there is a constant \( C \) such that

\[
\# C_l \leq C N 4^{-(L-l)}.
\]

Next we estimate the number of entries in the three matrices of (23) and (24) for one pair of neighboring cubes in level \( l \). The \( \Phi \) and \( \Psi \)-functions of a cube are transformations of all \( \Phi \)-functions in the cube’s children. The number of \( \Phi \)-functions in a cube is given by the dimension of \( \Pi_{p,L} \), which is

\[
n_l = (p_l + 1)(p_l + 2)(p_l + 3)/6.
\]

Since there are at most eight non-empty children, the number of entries in these three matrices is less than \( 64 n_{l+1}^2 \). Furthermore, there are at most nine neighbors, and hence there is a constant \( C \) such that

\[
\# \text{ Entries} \leq C N \sum_{l=L}^{2} 4^{-(L-l)} n_{l+1}^2 \leq C N \sum_{l=2}^{L} 4^{-(L-l)} (L-l)^6.
\]

Note that the factor \( p_L \) in (30) is absorbed in the constant, and hence the last sum is bounded independently of \( L \). The coarsest-level blocks in (25) contribute at most \( 64 n_L^2 \) entries, which is of lower order. Therefore the number of entries in the non-standard form is \( O(N) \).

Since the number of entries in \( Q_\nu \) is bounded by \( 64 n_L^2 \) a similar argument as in (31) shows that the complexity of a wavelet transform using the pyramid scheme is order \( N \).

4. **Analysis of the Truncation Error**

To derive estimates for the error due to truncating terms in the non-standard form we begin with bounds for the magnitude of well separated interactions.

For the uniform refinement scheme considered here there are uniform constants such that

\[
|S_\nu| \leq c 2^{-2l}, \quad \nu \in C_l,
\]

\[
|S_\nu| \leq c \max_{\bar{x} \in S_\nu} |x - \bar{x}|^2, \quad \bar{x} \in B_\nu, \quad \nu \in C_l,
\]

holds for \( 2 \leq l \leq L \).

**Theorem 4.1.** If the kernel of the integral operator \( K \) satisfies Assumption 3.1 and \( f_\nu \in W_\nu, g_\nu \in X_\nu \) or \( f_\nu \in X_\nu, g_\nu \in W_\nu \) then for \( \nu, \nu' \in C_l \)

\[
|\langle f_\nu, K g_\nu' \rangle| \leq C 2^{-l} \eta_{l,\nu'}^p \| f_\nu \|_0 \| g_\nu' \|_0
\]

holds.
Proof. We only prove the assertion for the case that \( g_{\nu'} \in W_{\nu} \). The other case is completely analogous.

Since \( g_{\nu'} \) is orthogonal to functions in \( \Pi_{p,\nu'} \), it follows that

\[
\langle f_{\nu}, Kg_{\nu'} \rangle = \int_{S_{\nu}} \int_{S_{\nu'}} \left( G(|x-y|) - T(y) \right) f_{\nu}(x) g_{\nu'}(y) \, dS_y \, dS_x
\]

for every function \( T \in \Pi_{p,\nu'} \). Therefore we can estimate

\[
|\langle f_{\nu}, Kg_{\nu'} \rangle| \leq \inf_{T \in \Pi_{p,\nu'}} \max_{y \in S_{\nu'}} \left| G(|x-y|) - T(y) \right| |S_{\nu'}|^{1/2} \| f_{\nu} \|_{L^2} \| g_{\nu'} \|_{L^2}.
\]

Replace \( T \) by the generalized Taylor polynomial (28) with some expansion point \( \bar{y} \in B_{\nu'} \). In view of Assumption 3.1 and (26) this leads to the estimate

\[
|\langle f_{\nu}, Kg_{\nu'} \rangle| \leq \sup_{\nu' \in \nu} \frac{|S_{\nu'}|^{1/2} \| f_{\nu} \|_{L^2} \| g_{\nu'} \|_{L^2}}{\max_{y \in S_{\nu'}} |y - \bar{y}|}.
\]

Combining this result with (32) and (33) shows the assertion. \( \square \)

Remark 4.2. If both \( f \) and \( g \) are wavelets, estimates (34) can be improved. However, this is not relevant for the analysis of the non-standard form.

To estimate the truncation error due to all cubes in one level we need the following result, whose proof can be found in [21].

**Lemma 4.3.** Let \( y = T_{p,d} x \), where \( T_{p,d} : l_2(\mathbb{Z}^3) \to l_2(\mathbb{Z}^3) \) is defined by

\[
y_i = \sum_{\ell \in \mathbb{Z}^3: |\ell| \geq d} \frac{x_{\ell}}{\ell} \quad i \in \mathbb{Z}^3,
\]

then there is a constant \( C \) such that

\[
\|T_{p,d}\| \leq Cd^{-p}, \quad p \geq 2,
\]

where the constant depends on \( d \) but not on \( p \).

Since the centers of the enclosing boxes do not form a regular lattice, this lemma cannot be applied directly. However, because of the subdivision scheme of the cubes, each cube \( \nu \) is associated with a lattice point \( \nu' \in \mathbb{Z}^3 \). Note that our notation does not distinguish between the cube and the lattice coordinate. We can estimate

\[
\max_{x \in S_{\nu}} |x - x_{\nu}| \leq \gamma_1 a^{2^{-l}} \quad \quad \quad |x_{\nu} - x_{\nu'}| \geq \gamma_2 a^{2^{-l}} |\nu - \nu'|, \quad \text{for} \quad |\nu - \nu'| \geq 2,
\]

where \( a \) is the sidelenath of the top-level cube. If all panels are contained in their cube, then \( \gamma_1 = \sqrt{3}/2 \) and \( \gamma_2 = 1/2 \). Because of the condition on the protrusion of \( S_{\nu} \) the constant \( \gamma_2 \) is uniformly bounded and \( \gamma_1 \) is uniformly bounded away from zero.

For the separation ratio we estimate

\[
\eta_{\nu,\nu'} \leq \min_{\ell_1 \in \ell_{\nu}, \ell_2 \in \ell_{\nu'}} \frac{|y - \bar{y}|}{|x - \bar{x}| + |y - \bar{y}|} \leq \min_{\ell_1 \in \ell_{\nu}, \ell_2 \in \ell_{\nu}} \frac{|x - \bar{x}| + |y - \bar{y}|}{|x - \bar{x}| + |y - \bar{y}|}.
\]

Thus we see that

\[
\eta_{\nu,\nu'} \leq \frac{|x - x_{\nu}| + |y - x_{\nu'}|}{|x_{\nu} - x_{\nu'}|} = \frac{\rho_{\nu} + \rho_{\nu'}}{|x_{\nu} - x_{\nu'}|}
\]
Combining this with the above estimates, it follows that

\[(37) \quad \eta_{\nu, \nu'} \leq \frac{\gamma}{|\nu - \nu'|}\]

holds for \(\gamma = \gamma_1/\gamma_2\) and \(|\nu - \nu'| \geq 2\).

**Lemma 4.4.** Let \(y = T_i x\), where \(T_i : I_2(C_i) \to I_2(C_i)\) is defined by

\[y_{\nu} = \sum_{\nu' \notin N(\nu)} \eta_{\nu', \nu}^{p_{\nu} + 1} x_{\nu'}, \quad \nu \in C_i\]

then

\[\|T_i\| \leq C_{\eta}^p, \quad p_k > 2.\]

holds uniformly in \(l\).

**Proof.** For the constants \(\gamma\) and \(\eta\) are defined in (37) and (27), let \(d\) be the smallest integer for which \(\gamma \leq \eta d\) holds. The decomposition

\[(38) \quad y_{\nu} = \sum_{\nu' \notin N(\nu), |\nu - \nu'| \leq d} \eta_{\nu, \nu'}^{p_{\nu} + 1} x_{\nu'} + \sum_{\nu' \notin N(\nu), |\nu - \nu'| > d} \eta_{\nu, \nu'}^{p_{\nu} + 1} x_{\nu'}\]

gives rise to two operators on \(I_2(C_i)\), denoted by \(T_i^a\) and \(T_i^f\). Because of

\[\sum_{\nu' \notin N(\nu), |\nu - \nu'| \leq d} \eta_{\nu, \nu'}^{p_{\nu} + 1} x_{\nu'} \leq \eta^{p_{\nu} + 1} \sum_{\nu' \notin N(\nu), |\nu - \nu'| \leq d} |x_{\nu'}|\]

and the fact that the number of terms in the sum is uniformly bounded it follows that \(\|T_i^a\| < C_{\eta}^p\). For the second term in (38) we set \(\eta_{\nu, \nu'} = 0\) if either \(\nu\) or \(\nu'\) correspond to an empty cube. Thus the operator \(T_i^f\) can be extended to act on \(I_2(\mathbb{Z}^3)\) by

\[\tilde{y}_{\nu} = \sum_{\nu' \notin N(\nu), |\nu - \nu'| > d} \eta_{\nu, \nu'}^{p_{\nu} + 1} x_{\nu'}, \quad \nu \in \mathbb{Z}^3.\]

From (37) the estimate

\[|\tilde{y}_{\nu}| \leq \sum_{\nu' \notin \mathbb{Z}^3, |\nu - \nu'| > d} \eta_{\nu, \nu'}^{p_{\nu} + 1} |x_{\nu'}| \leq \eta^{p_{\nu}} \sum_{\nu' \notin \mathbb{Z}^3, |\nu - \nu'| > d} \frac{|x_{\nu'}|}{|\nu - \nu'|^{p_{\nu} + 2}}\]

follows, hence, by Lemma 4.3 \(\|T_i^f\| \leq C(\gamma/d)^{p_k} \leq C_{\eta}^p\). The assertion follows from the triangle inequality. \(\square\)

**Theorem 4.5.** If \(\eta < 1/2\) and \(p_k\) is given by (30), then there is a constant \(C\) such that for \(f_h, g_h \in X_h\) the consistency error can be estimated by

\[(39) \quad \|\langle f_h, (K - K^c) g_h \rangle\| \leq C h \|g_h\|_0 \|f_h\|_0.\]

**Proof.** The error due to truncating well separated terms from the non-standard form is given by

\[(40) \quad \langle f_h, (K - K^c) g_h \rangle = \sum_{l=2}^L \sum_{\nu' \in C_l} \left( \langle \tilde{f}_{\nu}, K \tilde{g}_{\nu} \rangle + \langle \tilde{f}_{\nu}, K \tilde{g}_{\nu'} \rangle + \langle \tilde{f}_{\nu'}, K \tilde{g}_{\nu'} \rangle \right).\]
Since a bound for each term in this sum is given by Theorem 4.1, it remains to add up all the terms
\[
\| \langle f, (K - K^c) g \rangle \| \leq C \sum_{i=2}^{L} 2^{-i} \sum_{\nu' \in c_i} \eta_{\nu^i}^{P+1} \| \tilde{f}_{\nu'} \| \| \tilde{g}_{\nu'} \| + \| \tilde{f}_{\nu'} \| \| \tilde{g}_{\nu'} \| + \| \tilde{f}_{\nu'} \| \| \tilde{g}_{\nu'} \| 
\]
\[
\leq C \sum_{i=2}^{L} 2^{-i} \eta^{P+1} \| \tilde{f}_{\nu} \| \| \tilde{g}_{\nu} \| + \| \tilde{f}_{\nu} \| \| \tilde{g}_{\nu} \| + \| \tilde{f}_{\nu} \| \| \tilde{g}_{\nu} \| 
\]
\[
\leq C 2^{-L} \sum_{i=2}^{L} (2\eta)^{L-i} (L - 1) \| \tilde{f}_{\nu} \| \| \tilde{g}_{\nu} \| + \| \tilde{f}_{\nu} \| \| \tilde{g}_{\nu} \| + \| \tilde{f}_{\nu} \| \| \tilde{g}_{\nu} \| 
\]
\[
\leq C 2^{-L} \left\{ \gamma_L \left( \sum_{i} ||\tilde{f}_{\nu}||^2 \right) \right\}^{\frac{1}{2}} \left( \sum_{i} ||\tilde{g}_{\nu}||^2 \right) + \gamma_{\tilde{f}} \left( \sum_{i} ||\tilde{f}_{\nu}||^2 \right) \right\}^{\frac{1}{2}} 
\]
where
\[
\gamma_L = \left( \sum_{i=2}^{L} (2\eta)^{2(L-i)} (L - 1)^{2\kappa} \right) \frac{1}{2} \quad \text{and} \quad \tilde{\gamma}_L = \max_i (2\eta)^{L-i} (L - 1)^{\kappa}. 
\]
From the stability estimates (19) and (20) it follows that
\[
\| \langle f, (K - K^c) g \rangle \| \leq C 2^{-L} (2\gamma_L + \tilde{\gamma}_L) ||f|| \| \| g \| 
\]
Since $2\eta < 1$, the factors $\gamma_L$ and $\tilde{\gamma}_L$ are bounded independently of $L$. This proves the assertion.

**Truncation error.** Consider the boundary integral equation in variational form
\[
\langle g, A f \rangle = \langle g, u \rangle, \quad g \in L^2(S) 
\]
where $A = \lambda + K$ and $\lambda \neq 0$ is in the resolvent set of $K$. To solve this equation numerically apply the compressed Galerkin scheme
\[
\langle g_h, A^c f_h \rangle = \langle g_h, u \rangle, \quad g_h \in X_h, 
\]
where $A^c = \lambda + K^c$.

The next theorem gives an estimate of the error $e_h = f - f_h$. The proof relies on the consistency result of Theorem 4.5 and a well known stability argument. Nevertheless, we present a complete proof to make this article self consistent.

**Theorem 4.6.** The compressed Galerkin scheme is stable, i.e.,
\[
\| A^c g_h \| \leq C \| g_h \| , \quad g_h \in X_h 
\]
if $h$ is sufficiently small. Furthermore, if the solution $f$ belongs to $H^1(S)$ then the error is bounded by
\[
\| f - f_h \| \leq C h \| f \| . 
\]
Proof. Lemma 1 in Sec. 13.1 of [6] implies that

$$\|A g_h\|_0 \geq C \|g_h\|_0, \quad g_h \in X_h.$$  

Therefore, by the inverse triangle inequality and (39)

$$\|A^c g_h\|_0 \geq \|A g_h\|_0 - \|(A - A^c) g_h\|_0 \geq \|A g_h\|_0 - C h \|g_h\|_0 \geq C \|g_h\|_0$$

provided that $h$ is sufficiently small. This demonstrates (43).

Let $P_h$ be the orthogonal projector into $X_h$, then (39) can be rewritten as

$$\|(P_h A - A^c) f\|_0 \leq C h \|f\|_1.$$  

To prove the second part of the theorem, note that

$$A^c f_h = P_h A f.$$  

Using these results, (43), and standard approximation results of $P_h$ we estimate

$$\|P_h f - f_h\|_0 \leq C \|A^c (P_h f - f_h)\|_0$$

$$= C \|A^c P_h f - P_h A f\|_0$$

$$\leq C \|(A^c - P_h A) f\|_0 + \|P_h A - P_h f\|_0 \leq C h \|f\|_1$$

Finally, the assertion follows from the decomposition

$$\|f - f_h\|_0 \leq \|P_h f - f_h\|_0 + \|P_h f - f\|_0 \leq h \|f\|_1$$

$\Box$

5. Computation of Matrix Coefficients

The analysis in the preceding section is based on the assumption that the entries in the stiffness matrix have been calculated exactly. In realistic situations, numerical quadrature must be applied which must be done carefully to preserve the asymptotic property of the method.

This section discusses an algorithm that works in a similar manner as the pyramid scheme. First the interactions of standard basis functions in neighboring cubes of the finest level are calculated, then the interactions of $\Phi$- and $\Psi$-functions at the same level are computed as transformations of the interactions in the standard basis. In the coarser levels, interactions of $\Phi$- and $\Psi$-functions of neighboring cubes are computed by transforming the interactions between $\Phi$-functions of their children cubes. In this step interactions at the finer level will be needed that do not correspond to neighboring cubes and hence have not been computed before. These interactions are approximated using a generalized Taylor polynomial.

We will need a modified definition of the separation ratio of two cubes, namely

$$\tilde{\eta}_{\nu, \nu'} := \left\{ \begin{array}{ll}
\max_{\nu \in K(\nu)} \frac{\nu_{l+1} - \nu_{l}}{\nu_{l+1} - \nu_{l-1}} & \nu, \nu' \in C_L, \quad L - 1 \leq l \leq 2 \\
\max_{\nu \in K(\nu)} \frac{\nu_{l+1} - \nu_{l}}{\nu_{l+1} - \nu_{l-1}} & \nu, \nu' \in C_L.
\end{array} \right.$$  

In analogy, the modified neighbors are given by

$$N_1(\nu) = \{ \nu' \in C_L : \max \tilde{\eta}_{\nu, \nu'} > \eta \}$$

In order to describe the algorithm for the quadrature of the matrix entries the set of second-nearest neighbors will be needed, which is defined by

$$N_2(\nu) := \{ \nu' \in C_L : \text{parent}(\nu') \in N_1(\text{parent}(\nu)) \} ,$$
Furthermore, we set
\[ N_2^* (\nu) := N_2^*(\nu) \setminus N(\nu) . \]

From (35) it follows that
\[ \eta_{\nu, \nu'} \leq \tilde{\eta}_{\nu, \nu'} \]
and hence \( N(\nu) \subset N_1(\nu) \), but this fact does not affect the asymptotic estimates of the complexity.

**Finest level.** Calculate interactions of the standard basis functions contained in all pairs of second-nearest neighbors
\[ A_{\nu, \nu'} = (\langle \chi_i, K \chi_j \rangle)_{i \in N_{\nu}, j \in N_{\nu'}, \nu \in C_L, \nu' \in N_2(\nu)} . \]

These integrals are either singular or nearly singular, and are calculated with standard quadrature methods. The interaction matrices of \( \Phi \)- and \( \Psi \)-functions in the finest level are, because of (7) and (8), transformations of the interactions of the standard basis,
\[ \left[ \begin{array}{cc} \langle \Phi_{\nu}, K \Phi_{\nu'} \rangle & \langle \Phi_{\nu}, K \Psi_{\nu'} \rangle \\ \langle \Psi_{\nu}, K \Phi_{\nu'} \rangle & \langle \Psi_{\nu}, K \Psi_{\nu'} \rangle \end{array} \right] = Q_{\nu}^T A_{\nu, \nu'} Q_{\nu'} , \quad \nu \in C_L, \nu' \in N(\nu) . \]

Note that only the interaction between first and second neighbors are needed to set up the non-standard form. However, the matrices corresponding to neighboring cubes in the next coarser level depend on interactions of \( \Phi \)-functions supported by second-nearest neighbors at the finest level. Thus these interactions are also computed by using the transform
\[ \langle \Phi_{\nu}, K \Phi_{\nu'} \rangle = \tilde{Q}_{\nu}^T A_{\nu, \nu'} \tilde{Q}_{\nu'} , \quad \nu \in C_L, \nu' \in N_2^*(\nu) . \]

**Coarser levels.** We assume that interactions of \( \Phi \)-functions between neighbors and second neighbors
\[ A_{\nu, \nu'} = (\langle \Phi_{i}, K \Phi_{j} \rangle)_{i \in I_{\nu}, j \in I_{\nu'}, \nu \in C_l, \nu' \in N_2(\nu)} , \]
have been computed in level \( l+1 \). As in the finest level it follows that
\[ \left[ \begin{array}{cc} \langle \Phi_{\nu}, K \Phi_{\nu'} \rangle & \langle \Phi_{\nu}, K \Psi_{\nu'} \rangle \\ \langle \Psi_{\nu}, K \Phi_{\nu'} \rangle & \langle \Psi_{\nu}, K \Psi_{\nu'} \rangle \end{array} \right] = Q_{\nu}^T A_{\nu, \nu'} Q_{\nu'} , \quad \nu \in C_l, \nu' \in N(\nu) . \]

To set up matrices in the next coarser level, it is also necessary to compute the interactions between \( \Phi \)-functions of second neighbors. These are transformations of the children’s \( \Phi \)-functions as in (53). However, these interactions have not been calculated before. Since these interactions are well separated, they are approximated by replacing the kernel by a generalized Taylor polynomial
\[ G(x - y) \approx \sum_{|\beta + \gamma| \leq \tilde{p}_k} D^{\beta + \gamma} G(r_{\mu, \mu'}) \frac{h_{\beta, \mu}(x) h_{\gamma, \mu'}(y)}{\beta! \gamma!} \]
where \( r_{\mu, \mu'} = x_{\mu} - x_{\mu'} \). This leads to
\[ \langle \Phi_{i}, K \Phi_{j} \rangle \approx \sum_{|\alpha + \beta| \leq \tilde{p}_k} \frac{D^{\alpha + \beta} G(r_{\nu, \nu'})}{\alpha! \beta!} m^\alpha(\Phi_{i}) (\Phi_{j}) (-1)^{|\beta| \mu^\beta(\Phi_{j})} . \]

The expansion order \( \tilde{p}_k \) must be adjusted to the level to preserve the optimality of the scheme. In the following we will show that this is the case when
\[ \tilde{p}_l = \tilde{p}_L + L - l \]
where \( \tilde{p}_L \geq p_L \) is fixed as the mesh is refined and typically very small.

To evaluate (54) it suffices to know the moments of the \( \Phi \)-functions and the partial derivatives of the kernel up to order \( \tilde{p} \). The moments have been calculated when setting up the basis. The partial derivatives of the kernel can be computed for many relevant problems by recurrence formulas, see [17].

For a pair of cubes, the cost of evaluating all Taylor series in (54) and the cost to compute the transformation (52) is proportional to \( n_0^3 \). Thus an argument similar to (31) shows that the total complexity of these operations is \( O(N) \).

**Quadrature Error Analysis.** In this section we will demonstrate that the additional error due to the quadrature does not affect the asymptotic accuracy of the scheme. It is difficult to estimate the error due to quadrature in the non-standard form (22) because entries in coarse levels are computed recursively from entries in the fine levels. To rewrite the non-standard form into a form that shows the quadrature error more clearly, we begin with the following obvious identity

\[
\sum_{\nu, \nu' \in C_l} \langle f_\nu, K g_{\nu'} \rangle = \sum_{\nu, \nu' \in C_l} \langle \hat{f}_\nu, \hat{K} \hat{g}_{\nu'} \rangle + \sum_{l=2}^{L} \sum_{\nu, \nu' \in C_l} \langle \hat{f}_\nu, \hat{K} \hat{g}_{\nu'} \rangle + \langle \hat{f}_\nu, \hat{K} \hat{g}_{\nu'} \rangle + \langle \hat{f}_\nu, \hat{K} \hat{g}_{\nu'} \rangle
\]

where \( f_\nu = \hat{f}_\nu + f_\nu \) and \( g_{\nu'} = \hat{g}_{\nu'} + g_{\nu'} \). Using this fact, the truncated non-standard form can be rewritten as

\[
\langle f_h, K^c g_h \rangle = \sum_{\nu, \nu' \in C_l} \langle \hat{f}_\nu, \hat{K} \hat{g}_{\nu'} \rangle + \sum_{l=2}^{L} \sum_{\nu, \nu' \in C_l} \langle \hat{f}_\nu, \hat{K} \hat{g}_{\nu'} \rangle + \langle \hat{f}_\nu, \hat{K} \hat{g}_{\nu'} \rangle + \langle \hat{f}_\nu, \hat{K} \hat{g}_{\nu'} \rangle
\]

The same argument can be repeated for all finer levels, resulting in an alternative representation of the compressed non-standard form

\[
\langle f_h, K^c g_h \rangle = \sum_{l=2}^{L} \sum_{\nu, \nu' \in C_l} \langle \hat{f}_\nu, \hat{K} \hat{g}_{\nu'} \rangle + \sum_{\nu, \nu' \in C_L} \langle f_\nu, g_{\nu'} \rangle = \sum_{l=2}^{L} \sum_{\nu, \nu' \in C_l} \langle \hat{f}_\nu, \hat{K} \hat{g}_{\nu'} \rangle + \sum_{\nu, \nu' \in C_L} \langle f_\nu, g_{\nu'} \rangle
\]

Thus the quadrature error consists of two parts, namely an error due to replacing \( \mathcal{N}_2 \)-interactions by truncated Taylor series (first error) and an error due to replacing neighboring interactions in the finest level by a quadrature rule (second error)

\[
\langle f_h, (K^c - K^c_h) g_h \rangle = \sum_{l=2}^{L} \sum_{\nu, \nu' \in C_l} \langle f_\nu, (K - K^c_h) \hat{g}_{\nu'} \rangle + \sum_{\nu, \nu' \in C_L} \langle f_\nu, (K^c - K^c_h) g_{\nu'} \rangle
\]
It can be shown that the the nearfield in (56) is $O(h)$, thus it could be dropped without affecting the asymptotic error estimate. However, the constant is large and therefore it is beneficial to include the nearfield with some quadrature that reduces the constant. This can be accomplished, for instance, by using Duffy-like transformations in order $N$ complexity. For more details we refer to [14].

The first error can be estimated in a similar manner as the truncation error. Consider one term in the first sum of (57) and recall that the integral operator is replaced by a truncated Taylor series in the children of these cubes. Setting $f_{\mu} = f_{\nu} |_{S_{\mu}}$ and $g_{\mu'} = g_{\nu'} |_{S_{\mu'}}$, we obtain

$$
\langle \tilde{f}_{\nu}, (K^c - K^c_{\nu'}) \tilde{g}_{\mu'} \rangle = \sum_{\mu \in \mathcal{K}(\nu')} \langle f_{\mu}, (K^c - K^c_{\nu'}) g_{\mu'} \rangle
$$

An estimate for one of the terms in the above sum follows from Assumption 3.1 where the expansion center $\bar{y}$ is replaced by $x_{\mu'} - x_{\mu}$, thus

$$
|\langle f_{\mu}, (K^c - K^c_{\nu'}) g_{\mu'} \rangle| \leq C 2^{-l} \left( \frac{r_{\nu} + r_{\mu'}}{r_{\mu'}} \right) \tilde{p}_l ||f_{\mu}||_0 ||g_{\mu'}||_0 .
$$

Adding the contribution of all children together, we obtain in view of (45)

$$
|\langle f_{\nu}, (K^c - K^c_{\nu'}) g_{\nu'} \rangle| \leq C 2^{-l} \tilde{\eta}_{\nu, \nu'} ||f_{\nu}||_0 ||g_{\nu'}||_0 , \quad \nu' \in \mathcal{N}(\nu).
$$

This estimate will be necessary to prove the following result.

**Theorem 5.1.** If $\eta < 1/2$ and $\tilde{p}_l$ is given by (55) then there is a constant $C > 0$ such that for $f_h, g_h \in X_h$ the quadrature error can be estimated by

$$
|\langle f_{\nu}, (K^c - K^c_{\nu'}) g_{\nu'} \rangle| \leq C h ||g_h||_0 ||f_h||_0 .
$$

**Proof.** We add the contribution of all terms in (57). Since the number of cubes in $\mathcal{N}_2^c(\nu)$ is uniformly bounded, we can estimate, using similar arguments as in the proof Theorem 4.5

$$
\left| \sum_{l=2}^{L} \sum_{\nu' \in \mathcal{N}_2^c(\nu)} \langle f_{\nu}, (K^c - K^c_{\nu'}) g_{\nu'} \rangle \right| \leq C \max_{l, \nu \in C_l} |\mathcal{N}_2^c(\nu)| \sum_{l=2}^{L} 2^{-l} \tilde{\eta}_{\nu, \nu'} ||f_{l}||_0 ||\tilde{g}_{l}||_0 
$$

$$
\leq C 2^{-L} ||f_h||_0 ||g_h||_0 .
$$

Theorem 5.1 immediately implies that Theorem 4.5 also holds if the matrix coefficients are computed by the quadrature scheme. Therefore Theorem 4.6 also holds in the presence of quadrature.

6. **Computation of Transformation Matrices**

This section discusses two algorithms to set up the moment matrices $M(\chi_{\nu})$ and $M(\phi_{\nu})$ which are necessary for the computation of the transformation matrices $Q_{\nu}$ and the quadrature in (54). The first algorithm generates the moments corresponding to the exact monomials. This algorithm has a logarithmic factor in its complexity estimate. The second algorithm generates moments corresponding to approximate monomials in $O(N)$ operations. We will show that this approximation of the monomials does not affect the asymptotic convergence.
Exact Moments. We first consider the case where the functions \( h_{\nu, \alpha} \) in the definition of the moments in (5) are the monomials. The moment matrices can be computed in a recursive manner which is in essence the upward pass of the Fast Multipole Method. The matrices in the finest level \( M(\chi_{\nu}) \) are computed exactly. Suppose now that the \( M(\Phi_{\mu}) \)'s have been computed for all \( \mu \in C_{l+1} \). To compute \( M(\phi_{\nu}) \) for \( \nu \in C_l \), we need the moments in \( \nu \)'s children, but the center must be translated. If we write \( x - x_{\nu} = x_{\mu} - x_{\nu} + x - x_{\mu} \) and apply the binomial theorem we obtain

\[
m_{\nu, \alpha}(\Phi_{\mu, i}) = \int_{S_{\mu}} (x - x_{\nu})^\alpha \Phi_{\mu, i}(x) dS_x \\
= \sum_{\beta + \gamma = \alpha, \beta! \gamma!} \frac{(x_{\mu} - x_{\nu})^\beta}{\beta!} \frac{1}{\gamma!} \int_{S_{\mu}} (x - x_{\mu})^\gamma \Phi_{\mu, i}(x) dS_x \\
= \sum_{\beta + \gamma = \alpha, \beta! \gamma!} \frac{(x_{\mu} - x_{\nu})^\beta}{\beta!} \frac{m^\gamma_{\nu}(\Phi_{\mu, i})}{\gamma!}
\]

The problem is that the high-order moments in the coarse levels depend on moments of the same order in the finer levels. Thus all moments in all levels must be computed with the same order \( p_2 \) even if most of the moments will not be used otherwise. Therefore the total cost to compute all \( M(\Phi_{\nu}) \)'s is not optimal.

Approximate Moments. To obtain an optimal scheme we neglect the high-order moments in the fine levels.

In the finest level the moment matrices are computed only up to order \( p_2 \) using exact quadrature rules. Thus the functions in the definition of the moments (5) are monomials

\[ h_{\nu, \alpha}(x) = (x - x_{\nu})^\alpha, \quad \nu \in C_L. \]

The moments in the coarser level are computed using the formula

\[
m_{\nu, \alpha}^\alpha(\Phi_{\mu, i}) = \sum_{\beta + \gamma = \alpha, \beta! \gamma! \leq p_1} \frac{(x_{\mu} - x_{\nu})^\beta}{\beta!} \frac{m^\gamma_{\nu}(\Phi_{\mu, i})}{\gamma!}, \quad |\alpha| \leq p_0.
\]

This leads to an \( O(N) \) algorithm for the computation of all moments.

Let \( x \in S \), denote by \( \nu(l) \) the cube in level \( l \) that contains the point \( x \) and furthermore, let \( t_L = x - x_{\nu(l)} \) and \( t_k = x_{\nu(k+1)} - x_{\nu(k)} \). Because of the recursion (60) the moments of a function \( f_{\nu} \in X_{\nu} \) have the form

\[
m_{\nu}^\alpha(f_{\nu}) = \int_{S_{\nu}} h_{\alpha, \nu}(x) f_{\nu}(x) dS_x
\]

where

\[
h_{\alpha, \nu}(x) = \alpha! \sum_{\sum a_i x_i = \alpha} \frac{\partial^{a_\nu} f_{\nu}}{a_1!} \ldots \frac{x^a}{a_i!}, \quad \nu \in C_L, l \leq L.
\]

The functions \( h_{\alpha, \nu} \) are approximations to the monomials \((- x_{\nu})^\alpha\). In [16], it is shown that these functions satisfy Assumption 3.1, and hence the conclusions of Theorems 4.5, 5.1 and 4.6 also apply if the wavelet basis is generated with approximate moments.
7. Numerical Example

We illustrate the complexity and convergence of the truncated Galerkin scheme on an integral formulation related to Laplace’s equation. We consider the equation

\[ 2\pi \sigma(x) + \int_S \frac{\partial}{\partial n_x} \frac{1}{|x - y|} \sigma(y) \, dS_y + \int_S \frac{1}{|x_0 - y|} \sigma(y) \, dS_y = 4\pi, \quad x \in S, \]

where $S$ is the ellipsoid $(x_1/2)^2 + x_2^2 + (x_3/3)^2 = 1$ and $x_0$ is a point in the interior of the ellipsoid. The solution $\sigma$ is the equilibrium charge distribution when the ellipsoid has constant potential, [19]; and can be expressed analytically using ellipsoidal coordinates. Note that the integral operator in (61) is the adjoint of the double layer operator, but everything that has been said about the double layer operator can easily be applied to its adjoint.

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<td>7</td>
<td>4.822-10^8</td>
<td>5395.74</td>
<td>4578</td>
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</tr>
<tr>
<td>786432</td>
<td>8</td>
<td>1.543-10^9</td>
<td>17287.6</td>
<td>13655</td>
<td>0.00489</td>
</tr>
</tbody>
</table>

Table 1. Results for the ellipsoid

The timings in Table 1 refer to the total cpu-time which consists of the time to set up the basis, the quadrature for the entries in the non-standard form and the solution of the linear system. In all computations the quadrature accounts for more than 90 percent of the overall time. The linear systems are solved with GMRES, which converged in 9 to 11 iterations. In an optimal scheme, time and memory allocation are increased by a factor of four whereas the error is reduced by a factor of two. Our results closely reproduce this behavior.

All computations are in core on a single processor of a Compaq ES40 Alpha server with 833MHz clock speed. For more numerical experiments with the wavelet approach and a comparison with the Fast Multipole Method we refer to [18].

References


[18] ______, Sparse BEM for potential theory and Stokes flow using variable order wavelets, Computational Mechanics 32 (2003), no. 4-6, 312-319.


