Rapid Solution of Stokes Flow using Multiscale Galerkin BEM

Two issues that arise when solving Stokes flow problems with the hydrodynamical single-layer potential are addressed. First, the resulting boundary integral equation is singular, and second, discretizations lead to dense matrices. We discuss a well-posed modified equation which is equivalent for zero net-flux. Furthermore, we describe a multiscale basis that lead to sparse stiffness matrices. This approach is suitable for complicated geometries and is an extension of our previous work for the Laplace equation.

1. Introduction

The boundary element method is particularly effective for solving exterior Stokes flow problems in complicated, multiply-connected 3D geometries. Traditionally, integral formulations of this problem were based on the hydrodynamical double-layer operator [2], but more recently, the single-layer operator has received a great deal of interest in engineering applications, see e.g., [5]. The hydrodynamical single-layer operator is similar to its counterpart from potential theory in that its components have a $1/r^2$-behavior. Hence many issues of their numerical treatment are the same: For the piecewise constant Galerkin discretization the stiffness matrix is dense if standard box functions are used as a basis. Furthermore, the condition of the stiffness matrix grows like the inverse of the meshsize.

In the case of the Laplace equation we have described a multiscale basis that lead to an almost sparse matrix, i.e., a matrix of which most entries can be truncated without changing the asymptotic convergence of the discretization scheme [3]. Moreover, the system can be preconditioned to have a bounded condition as the mesh is refined. Contrary to the earlier wavelet-based approaches (see, e.g., [1, 4]), this basis is not defined in a parameter space of the surface. Therefore our construction leads to almost sparse matrices even for complicated and multiply connected geometries.

In this article we will demonstrate that this approach can be extended and is effective to treat Stokes flow problems. However, special care must be taken to avoid difficulties due to the fact that the hydrodynamical single-layer operator has a non-trivial nullspace. In Section 2 we describe how this singularity can be removed and in Section 3 we briefly discuss the multiscale basis. Finally, we conclude in Section 4 with numerical examples.

2. Integral Formulation

When the velocity field is expressed in terms of the hydrodynamical single-layer potential, then, assuming no-slip boundary conditions, the boundary integral equation

$$ V f(x) := \int_S G(x, y) f(y) dS_y = u(x), \quad x \in S, \quad (1) $$

arises. Here, $u(x) \in \mathbb{R}^3$ is a given velocity on the surface $S$, $f(x) \in \mathbb{R}^3$ is the unknown surface traction and $G(x, y) \in \mathbb{R}^{3 \times 3}$ is the Green’s function corresponding to Stokes flow

$$ G(x, y) = \frac{1}{8\pi} \left( \frac{1}{|x - y|} I + \frac{(x - y)(x - y)^T}{|x - y|^3} \right). \quad (2) $$

Since the velocity is divergence-free, the net-flux across each connected component $S_k, 1 \leq k \leq K$, of the surface must vanish. Thus the range of the hydrodynamical single-layer operator is $[L_2(S)]^3$-orthogonal to the $K$-dimensional subspace

$$ N := \text{span} \{ \chi_1 n, \ldots, \chi_K n \}, \quad (3) $$

where $n$ denotes the surface normal and $\chi_k$ the characteristic function of $S_k$. Since $V$ is self-adjoint, $N$ is also the nullspace of $V$. 

The single layer equation (1) is elliptic in the space $\tilde{H}^{-1/2}$, defined by

$$\tilde{H}^t = \left\{ f = (f_1, f_2, f_3)^T : f_i \in H^t(S), \int_{S_k} n^T f = 0 \right\}.$$  \hspace{1cm} (4)

To avoid having to deal with multiscale bases that span subspaces of $\tilde{H}^{-1/2}$, we modify (1) to an equivalent equation which is elliptic in $[H^{-1/2}(S)]^3$. To this end, note that the orthogonal projector into $N$ is given by

$$Pf(x) := \sum_{k=1}^K \chi_k(x)n(x) \frac{1}{|S_k|} \int_{S_k} n^T f.$$  \hspace{1cm} (5)

Consider the following modification of (1)

$$(V + P)f(x) = u(x).$$  \hspace{1cm} (6)

For the physically relevant right hand side $u \in \tilde{H}^{1/2}$, this equation is equivalent to equation (1), which is made precise in the following lemma.

Lemma 1. The bilinear form $a(f, g) := (f, (V + P)g)$ is elliptic in $[H^{-1/2}(S)]^3$. For given $u \in \tilde{H}^{1/2}$ equations (1) and (6) have the same solution $f \in \tilde{H}^{-1/2}$.

Proof. Let $X := \tilde{H}^{-1/2} \times N$ be equipped with the norm $\|(f^+, \tilde{f})\|^2 := \|f^+\|_{-1/2}^2 + \|\tilde{f}\|_0^2$, and let $Y := [H^{-1/2}(S)]^3$ be equipped with the usual Sobolev norm. The map $Id : Y \rightarrow X$ is defined through the orthogonal projection $P$ in (5) by $Id f := ((I - P)f, Pf)$. Then $Id$ is one-to-one and onto and its inverse is given by $Id^{-1}(f^+, \tilde{f}) = f^+ + \tilde{f}$. Furthermore, since $N$ is finite-dimensional, $Id^{-1}$ is bounded, and hence $Id$ is also bounded. Thus there are constants $c, C > 0$ such that for $f \in [H^{-1/2}(S)]^3$, $f^+ = (I - P)f$ and $\tilde{f} = Pf$ the norm equivalence

$$c \|f\|_{-1/2}^2 \leq \|f^+\|_{-1/2}^2 + \|\tilde{f}\|_0^2 \leq C \|f\|_{-1/2}^2$$

holds. By orthogonality of $\tilde{H}^{-1/2}$ and $N$ we also have $a(f, g) = (f^+, Vg^+) + (\tilde{f}, \bar{g})$. Since $(f^+, Vg^+)$ is elliptic in $\tilde{H}^{-1/2}$, we can now estimate

$$a(f, f) \geq c_1 \|f^+\|_{-1/2}^2 + \|\tilde{f}\|_0^2 \geq c_2 \|f\|_{-1/2}^2,$$

Moreover

$$a(f, g) \leq C_1 \|f^+\|_{-1/2}^2 \|g^+\|_{-1/2} + \|\tilde{f}\|_0 \|\bar{g}\|_0$$

$$\leq C_1 \left( \|f^+\|_{-1/2}^2 + \|\tilde{f}\|_0^2 \right)^{1/2} \left( \|g^+\|_{-1/2}^2 + \|\bar{g}\|_0^2 \right)^{1/2}$$

$$\leq C_2 \|f\|_{-1/2} \|g\|_{-1/2}.$$  \hspace{1cm}

This proves that $a(f, g)$ is elliptic in $[H^{-1/2}(S)]^3$. Finally, if $f$ is the solution of (6) for $u \in \tilde{H}^{1/2}$, then $(Vf - u) + Pf = 0$. Since $\tilde{H}^{1/2}$ and $N$ are orthogonal it follows that both terms in this sum must vanish. Therefore $f \in \tilde{H}^{-1/2}$ is also the solution of (1).

3. Multiscale Galerkin BEM

In this section we briefly describe the construction of the multiscale basis that lead to nearly sparse stiffness matrices. First the surface $S$ is embedded in a top-level cube $C_0$. The cube is subsequently subdivided into eight cubes of equal size and this process is iterated until the cubes in the finest level $L$ contain at most a predetermined number of panels of the surface triangulation $T$. The cubes that have a non-empty intersection with $S$ at the $l$-th refinement level are collectively denoted by $C_l, 0 \leq l \leq L$. The centroid of a cube $\nu \in C_l$ is denoted by $x_{\nu}$. Similar to wavelets, we define a nested sequence of singlescale spaces, the multiscale spaces are then the differences of the singlescale spaces.

For the single-scale spaces, consider first the following subspaces of the finite element space $X_h$

$$X_\nu := \{ \phi \in X_h : \phi(\nu) = 0 \text{ if } v \not\in \nu \text{ is the centroid of } \tau \in T \}.$$  \hspace{1cm} (7)
The support of functions in $X_\nu$ is denoted by $S_\nu$ and the polynomials in three variables up to degree $p$ are denoted by

$$\Pi_p = \text{span} \{ x^\alpha : |\alpha| \leq p \}. \quad (8)$$

The single scale space $V_\nu, \nu \in C_I$ is the $L_2(S)$-orthogonal projection of $\Pi_p$ into $X_\nu$. The single scale space of level $l$ is defined as

$$V^{(l)} = \bigoplus_{\nu \in C^{(l)}} V_\nu. \quad (9)$$

The multiscale spaces are the $L_2(S)$-orthogonal complements of the single-scale spaces at two consecutive levels

$$V^{(l+1)} = V^{(l)} \oplus W^{(l)}, \text{ for } L > l \geq 2, \quad (10)$$

and

$$X_h = V^{(L)} \oplus W^{(L)}. \quad (11)$$

Level two is the coarsest level where compression of the integral operator occurs. This construction leads to the multiresolution decomposition of $X_h$

$$X_h = V^{(2)} \oplus W^{(2)} \oplus \ldots \oplus W^{(L)}. \quad (12)$$

The moments of the function $f_\nu \in X_\nu$ are defined by

$$\mu^\alpha(f_\nu) = \int_{S_\nu} (x - x_\nu)^\alpha f(x) \, ds_x \quad (13)$$

where $x_\nu$ is the centroid of cube $\nu$ and $\alpha$ is a multi-index. The moments can be used to characterize functions in $W_\nu$:

$$\text{if } f \in V^{(l+1)} \text{ then } f \in W^{(l)} \iff \mu^\alpha(f|_{S_\nu}) = 0, |\alpha| \leq p, \nu \in C^{(l)}. \quad (14)$$

The vanishing moment property of the basis functions implies that entries in the stiffness matrix corresponding to functions with separated supports are small. In that situation, each component of the Green’s function in (2) can be Taylor expanded, which leads to an expression of a coefficient of the stiffness matrix in terms of the moments. For two functions $f_\nu \in X_\nu, f_{\nu'} \in X_{\nu'}$, where $S_\nu \cap S_{\nu'} = \emptyset$ the Taylor series converges and one has for $1 \leq i, j \leq 3$

$$\langle f_{\nu'}, V_{ij} f_\nu \rangle = \int_{S_\nu} \int_{S_{\nu'}} G_{ij}(r + h - k)f_\nu(x)f_{\nu'}(y) \, ds_x ds_y \quad (15)$$

$$= \sum_{\alpha} \frac{1}{\alpha!} \partial^\alpha G_{ij}(r) \int_{S_\nu} \int_{S_{\nu'}} (h - k)^\alpha f_\nu(x)f_{\nu'}(y) \, ds_x ds_y \quad (16)$$

$$= \sum_{|\alpha| \leq p} \partial^\alpha G_{ij}(r) \sum_{\beta \leq \alpha} (-1)^{|\beta|} \frac{(\alpha - \beta)!}{(\alpha - \beta)!} \int_{S_\nu} \int_{S_{\nu'}} h^{(\alpha - \beta)} f_\nu(x) \, ds_x \int_{S_{\nu'}} k^\beta f_{\nu'}(y) \, ds_y \quad (17)$$

$$= \sum_{\alpha} \partial^\alpha G_{ij}(r) \sum_{\beta \leq \alpha} (-1)^{|\beta|} \frac{(\alpha - \beta)!}{(\alpha - \beta)!} \mu^{\alpha - \beta}(f_\nu) \mu^\beta(f_{\nu'}). \quad (18)$$

If $f_{\nu'} \in W_{\nu'}$, then the terms for $|\alpha| \leq p$ in the above summation vanish; if in addition $f_{\nu'} \in W_{\nu'}$, then the terms for $|\alpha| \leq 2p$ vanish. Thus the error due to truncating matrix entries can be controlled by the separation of the truncated entries as well as the order $p$ in (8).

## 4. Numerical Results

Here we present numerical results for a translating sphere in an infinite fluid. For this problem the analytical solution is known. Our calculations are based on the non-standard form where matrix entries corresponding to cubes which do not share vertices are truncated. Thus the accuracy is only controlled by $p$. To solve the linear system we have used conjugate gradients (cg) and preconditioned cg (pcg). The iteration is stopped when the residual is reduced by $10^{-9}$. The preconditioner is the block diagonal matrix which consists of three blocks corresponding to the coarsest level single-scale space

$$P_{i,j} = \langle (\Phi_i, V_{ij} \Phi_j) \rangle_{1 \leq k, l \leq \text{dim} V^{(2)}}, \quad 1 \leq i \leq 3$$
and blocks corresponding to the multi-scale spaces
\[ P_{i,i,\nu} = \langle (\Psi_{\nu,k}, V_{\nu,l}) \rangle_{1 \leq k, l \leq \text{dim} W_{\nu}}, \quad 1 \leq i \leq 3, \nu \in C^{(i)}, 2 \leq l \leq L \] (19)

where \( \Phi_k \) are basis functions of \( V^{(2)} \) and \( \Psi_{\nu,k} \) are basis functions of \( W_{\nu} \).

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Table 1: \( L_2 \) discretization errors, translating sphere

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Table 2: iteration counts for translating sphere

From the table it is clear that \( p \) must be increased when the meshwidth is refined. Incrementing \( p \) for each refinement is sufficient to maintain the asymptotic convergence of the piecewise constant Galerkin method. In that case the number of pcg iterations to solve the system remains almost constant.

5. Conclusion

All observations on the convergence behavior of the truncated Galerkin scheme and the preconditioner in this article are purely empirical. We will present a detailed convergence analysis in a forthcoming paper.

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6. References


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