Choice under Uncertainty.

Decision maker chooses between risky alternatives that are probability distributions over outcomes.

Choice is made before uncertainty is resolved (outcomes are actually realized).

Set of all possible outcomes: C.

Outcomes may be consumption bundle or a monetary payoffs...

They are realized after all uncertainty is resolved.

No uncertainty in the outcomes.

Assume: *C is finite with N elements indexed $n = 1, ...N$.*

A simple lottery L is a probability distribution on the set of outcomes C i.e., a vector $(p_1, ...p_N)$,

$$p_n \geq 0, \sum_{n=1}^{N} p_n = 1,$$

where p_n denotes the probability of outcome n.

Each simple lottery is a vector or point in the $(N-1)$ dimensional simplex

$$\triangle = \{(p_1, ...p_N) \in \mathbb{R}_+^N : \sum_{n=1}^{N} p_n = 1\}.$$

A compound lottery is a probability distribution over simple lotteries.

Given K simple lotteries $L_1, ..., L_K$ and probabilities $(\alpha_1, ..., \alpha_K) \geq 0$, $\sum_{k=1}^{K} \alpha_k = 1$, a compound lottery $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$ is a risky alternative that yields the simple lottery L_k with probability α_k.

Every compound lottery can be reduced to an equivalent simple lottery in the sense that they both generate the same distribution over outcomes in C.

Consider simple lotteries $L_k = (p_1^k, ...p_N^k), k = 1, ...K$.

In the compound lottery $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$, the probability that any outcome $n \in C$ is realized is given by

$$\begin{aligned} & \sum_{k=1}^{K} \Pr\{\text{simple lottery } L_k \text{ is realized}\} \Pr\{\text{outcome } n \\ & \text{realized in } L_k\} \\ = & \sum_{k=1}^{K} \alpha_k p_n^k \end{aligned}$$

and so this compound lottery generates the same distribution over C as the simple lottery $L = (p_1, ...p_N)$ where

$$p_n = \alpha_1 p_n^1 + ... + \alpha_K p_n^K, n = 1, ...N.$$

Note that this reduced simple lottery can be obtained by vector addition:

$$
\begin{aligned}
L &= \alpha_1(p_1^1, ...p_N^1) + ... + \alpha_K(p_1^K, ...p_N^K) \\
&= \alpha_1 L_1 + ...\alpha_K L_K.
\end{aligned}
$$

Assume: decision maker cares only about the ultimate probability distribution over outcomes and therefore, for any compound lottery, only the reduced simple lottery is of relevance to the decision maker.

Set of risky alternatives for the decision maker: $\mathcal{L}$ where

$\mathcal{L} =$ *set of all simple lotteries over the set of outcomes* C.

This is the $(N-1)$ dimensional simplex.

Preferences:

Assume that the decision maker has a binary preference *ordering* $\succsim$ defined on $\mathcal{L}$.

Strict preference and indifference relations: $\succ, \sim$.

Two important axioms.

Continuity: *For any* $L, L', L" \in \mathcal{L}$, *the sets*

$$\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succsim L"\}$$

and

$$\{\alpha \in [0,1] : L" \succsim \alpha L + (1-\alpha)L'\}$$

are closed.

Equivalently, if a sequence of numbers $\{\alpha_i\}_{i=1}^{\infty} \to \alpha$, (where $\forall i, \alpha_i \in [0,1]$) and $\alpha_i L + (1-\alpha_i)L' \succsim L", \forall i$, then $\alpha L + (1-\alpha)L' \succsim L$ and if $L" \succsim \alpha_i L + (1-\alpha_i)L', \forall i$, then $L" \succsim \alpha L + (1-\alpha)L'$.

Lexicographic preferences violate continuity.

Example: Suppose C has three outcomes {An Ice-cream; A million dollars; Terrible Accident}.

Let $L" = (1,0,0), L = (0,1,0), L'(0,0,1)$.

Choose $\{\alpha_i\} \to \alpha = 1, 0 < \alpha_i < 1$.

Safety first preference: $L" \succ \alpha_i L + (1-\alpha_i)L' = (0, \alpha_i, 1-\alpha_i)$.

$\alpha_i L + (1-\alpha_i)L' \to L$

It is quite likely that $L \succ L"$.

Independence: *For all* $L, L', L" \in \mathcal{L}$ *and* $\alpha \in (0,1)$,

$$
\begin{aligned}
L &\succsim L' \\
&\iff \alpha L + (1-\alpha)L" \succsim \alpha L' + (1-\alpha)L".
\end{aligned}
$$

If we mix two lotteries with a third lottery, then the preference ordering of the two mixed lotteries remain unchanged i.e., is independent of the third lottery used.

Consider two states of nature "$rain$" and "$sunny$" that are realized with probabilities α and $1-\alpha$.

Think of the mixed lottery $\alpha L+(1-\alpha)L$" as the reduced form of the compound lottery $(L, L"; \alpha, 1-\alpha)$ i.e., the lottery that gives the agent the lottery L when it rains and lottery L" when it is sunny,

Likewise the mixed lottery $\alpha L'+(1-\alpha)L$" is the reduced form of the compound lottery $(L', L"; \alpha, 1-\alpha)$ i.e., the lottery that gives the agent the lottery L' when it rains and lottery L" when it is sunny.

The two compound lotteries yield the same lottery in the state of nature where it is sunny and differ only in the other state.

The independence axiom says the preference between these two compound lotteries (or their reduced forms) should depend only on L and L'; it should be independent of L" - if L" is replaced by some other lottery, the ordering of the two mixed lotteries must remain the same.

Argument: as long as the lotteries faced when the state is sunny are identical, the comparison of the two mixed lotteries ought to depend only on what happens when the state is rainy and in that event, *what might have happened if the state was sunny instead of rainy* should not matter.

Standard theory of consumer behavior in choices between bundles of goods: independence is a severe restriction.

For example, suppose there are three goods (coffee, tea, sugar) and the consumer cannot have coffee without sugar (complementarity) though she has tea without sugar.

Also the consumer likes coffee more than tea.

Consider bundles

$x = (2, 0, 0), y = (0, 2, 0), z = (0, 0, 2).$

The bundle $y \succ x$.

The bundle $\frac{1}{2}x + \frac{1}{2}z = (1,0,1) \succ \frac{1}{2}y + \frac{1}{2}z = (0,1,1)$

If z is replaced by $z' = (0,0,0)$ in the above mix, the ordering is reversed.

In this example, (part of) the bundle z is consumed with (part of) the bundles x or y.

However, in decision making under uncertainty, when one looks at a mixed lottery $\alpha L + (1-\alpha)L$", the lottery L" is not consumed with the lottery L because in the state in which L" is realized, L is not realized and vice-versa

(after the realization of initial uncertainty about which lottery is consumed, you consume L instead of L" or L" instead of L).

So, the usual argument for dependence in the standard consumer theory is not relevant.

Nonetheless, independence is the most controversial and strong restriction in the theory of decision making under uncertainty.

It is also the most important axiom needed for the expected utility theorem.

Representation:

A utility function $U : \mathcal{L} \to \mathbb{R}$ represents the preference ordering $\succsim$ if for every $L, L' \in \mathcal{L}$

$$L \succsim L' \Longleftrightarrow U(L) \geq U(L').$$

Can be shown that if $U : \mathcal{L} \to \mathbb{R}$ represents the preference ordering $\succsim$, then

$$\begin{aligned} L \ &\succ \ L' \Longleftrightarrow U(L) > U(L'). \\ L \ &\sim \ L' \Longleftrightarrow U(L) = U(L'). \end{aligned}$$

Mathematically, $\mathcal{L}$ is just like a finite dimensional commodity space with $N - tuples$ of numbers.

Just as in standard consumer theory, the continuity axiom is sufficient to ensure the existence of a utility function that represents the preference ordering $\succsim$.

In fact, there is a continuous utility function.

Every positive monotonic transformation of the utility function also represents $\succsim$.

von-Neumann and Morgenstern: Expected utility form.

Economists and decision theorists have long been interested in a specific kind of utility representation which is linear.

The utility function $U : \mathcal{L} \rightarrow \mathbb{R}$ *has an* expected utility form *if there are fixed numbers* $(u_1, ..., u_N)$ *so that for every lottery* $L = (p_1, p_2, .., p_N) \in \mathcal{L}$,

$$U(L) = u_1 p_1 + ... + u_N p_N.$$

i.e., $U(L)$ is a linear function of the probabilities with fixed coefficients $(u_1, ..., u_N)$.

Note that if $L = (0, 0, .., 1, ...0)$ where the probability one is assigned to the $i - th$ outcome (i.e., L is the degenerate lottery that realizes outcome i with certainty), then such an expected utility form would imply, $U(L) = u_i$.

So, one can interpret the coefficients u_i as the "utility" of the deterministic outcome i.

Thus, $u_1p_1 + ... + u_Np_N$ is the mathematical expectation (or, weighted average) of the utility that can be attained through the lottery L.

A utility function $U : \mathcal{L} \rightarrow \mathbb{R}$ with the expected utility form is called a *von Neumann-Morgenstern (vNM) expected utility function.*

Proposition. *A utility function* $U : \mathcal{L} \to \mathbb{R}$ *has an expected utility form* $\Longleftrightarrow$ *it is linear in the sense that for any* K *lotteries* $(L_1, ...L_K)$ *and for any* $K-$*vector of probabilities* $(\alpha_1, ...\alpha_K), \alpha_k \geq 0, \sum_{k=1}^{K} \alpha_k = 1$, *if* $L = \alpha_1 L_1 + ... + \alpha_K L_K$, *then*

$$U(L) = \alpha_1 U(L_1) + ... + \alpha_K U(L_K).$$

Proof. Suppose U has an expected utility form as described earlier. Let $L_k = (p_1^k, ..., p_N^k), k = 1, ...K$.

The lottery $L == (p_1, ...p_N)$ where

$$p_n = \alpha_1 p_n^1 + ... + \alpha_K p_n^K, n = 1, ...N.$$

Therefore,

$$\begin{aligned} U(L) &= \sum_{n=1}^{N} p_n u_n \\ &= \sum_{n=1}^{N} (\alpha_1 p_n^1 + ... + \alpha_K p_n^K) u_n \\ &= \sum_{n=1}^{N} \alpha_1 p_n^1 u_n + ... + \sum_{n=1}^{N} \alpha_K p_n^K u_n \\ &= \alpha_1 U(L_1) + ... + \alpha_K U(L_K). \end{aligned}$$

Suppose that U is linear in the above sense.

For $n = 1, ...N$, let $\widehat{L}_n = (0, ...1, ...0)$ be the specific degenerate lottery where probability 1 is on outcome n.

Fix $u_n = U(\widehat{L}_n), .n = 1, ..N$.

Then, for any lottery $L = (p_1, ...p_N) \in \mathcal{L}$, one can write

$$L = p_1 \widehat{L}_1 + ... + p_N \widehat{L}_N$$

and using linearity of U,

$$\begin{aligned}
& U(L) \\
= \; & p_1 U(\widehat{L}_1) + ... + p_N U(\widehat{L}_N) \\
= \; & u_1 p_1 + ... + u_N p_N.
\end{aligned}$$

The proof is complete.

The expected utility property of the utility function is not necessarily preserved under any positive monotonic transformation.

Suppose $U : \mathcal{L} \to \mathbb{R}$ is a utility function that represents $\succeq$ and has the expected utility form.

If $f : \mathbb{R} \to \mathbb{R}$ is any strictly increasing function then $W = f(U) : \mathcal{L} \to \mathbb{R}$ also represents $\succeq$, but W need not have the expected utility form (could be non-linear).

The expected utility property is not an ordinal property of utility functions on the space of lotteries - it is a cardinal property.

It is preserved only under $linear$ transformations.

Proposition. *Suppose that $U : \mathcal{L} \rightarrow \mathbb{R}$ is a vNM expected utility function representing the preference ordering $\succsim$ on $\mathcal{L}$. Then $W : \mathcal{L} \rightarrow \mathbb{R}$ is another vNM expected utility function representing the preference ordering $\succsim$ on $\mathcal{L}$* if and only if *there exists scalars $\beta > 0$ and γ such that*

$$W(L) = \gamma + \beta U(L), \forall L \in \mathcal{L}.$$

Proof. We will show the "if" part.

Suppose there exist scalars $\beta > 0$ and γ such that $W(L) = \gamma + \beta U(L), \forall L \in \mathcal{L}$.

We will show that W has the expected utility property.

Since $U : \mathcal{L} \rightarrow \mathbb{R}$ is a vNM expected utility function, there exists $(u_1, ..., u_N)$ such that for any lottery $L = (p_1, p_2, .., p_N) \in \mathcal{L}$,

$$U(L) = u_1 p_1 + ... + u_N p_N$$

Define a fixed set of constants $(w_1, ...w_N)$ where $w_n = (\gamma + \beta u_n), n = 1, ...N$.

Then, for any lottery $L = (p_1, p_2, .., p_N) \in \mathcal{L}$

$$\begin{aligned} W(L) &= \gamma + \beta U(L) \\ &= \gamma + \beta(u_1 p_1 + ... + u_N p_N) \\ &= \gamma(p_1 + p_2 + .. + p_N) + \beta(u_1 p_1 + ... + u_N p_N) \\ &= (\gamma + \beta u_1)p_1 + (\gamma + \beta u_2)p_2 + ... + (\gamma + \beta u_N)p_N \\ &= w_1 p_1 + ... + w_n p_n. \end{aligned}$$

Thus, W is a vNM expected utility function. //

Proposition. *If the preference relation $\succeq$ on $\mathcal{L}$ is represented by a utility function U that has the expected utility form, then $\succeq$ satisfies the continuity and independence axioms.*

Proof. First we show continuity. Consider any sequence $\{\alpha_i\}_{i=1}^{\infty} \to \alpha$, (where $\forall i, \alpha_i \in [0,1]$) and $\alpha_i L + (1 - \alpha_i)L' \succsim L", \forall i$.

Then,

$$U(\alpha_i L + (1-\alpha_i)L') \geq U(L"), \forall i,$$

and using the linearity of U

$$\alpha_i U(L) + (1-\alpha_i)U(L') \geq U(L"), \forall i,$$

which implies (taking limit as $i \longrightarrow \infty$)

$$\alpha U(L) + (1-\alpha)U(L') \geq U(L")$$

so that $\alpha L + (1-\alpha)L' \succsim L$.

Next, we show independence.

Consider $L, L', L" \in \mathcal{L}$ *and* $\alpha \in (0,1)$.

Need to show: $L \succsim L' \Longleftrightarrow \alpha L + (1-\alpha)L" \succsim \alpha L' + (1-\alpha)L"$.

Suppose $L \succsim L'$.

Then, $U(L) \geq U(L')$ so that

$$\alpha U(L) + (1-\alpha)U(L") \geq \alpha U(L') + (1-\alpha)U(L")$$

which implies

$$\alpha L + (1-\alpha)L" \succsim \alpha L' + (1-\alpha)L".$$

Suppose that $\alpha L + (1-\alpha)L" \succsim \alpha L' + (1-\alpha)L"$.

Then,

$$U(\alpha L + (1-\alpha)L") \geq U(\alpha L' + (1-\alpha)L")$$

and using linearity of U,

$$\alpha U(L) + (1-\alpha)U(L") \geq \alpha U(L') + (1-\alpha)U(L")$$

which implies that $U(L) \geq U(L')$.

Expected Utility Theorem.

Proposition. Suppose that the preference ordering $\succsim$ on $\mathcal{L}$ satisfies the continuity and independence axioms. There $\succsim$ admits a utility representation of the expected utility form.

Allais paradox.

$N = 3$.

$C = \{\$2.5 \text{ million}, \$0.5 \text{ million}, \$0\}$

Consider $L_1 = (0, 1, 0)$,

$L'_1 = (0.10, 0.89, 0.01)$.

$L_2 = (0, 0.11, 0.89)$,

$L'_2 = (0.10, 0, 0.90)$.

$Observed$:

$$L_1 = (0, 1, 0) \succ L'_1 = (0.10, 0.89, 0.01)$$

implies

$$U(0.5) > 0.1U(2.5) + 0.89U(0.5) + 0.01U(0)$$

which implies

$$0.11U(0.5) > 0.1U(2.5) + 0.01U(0)$$

so that

$$0.11U(0.5) + 0.89U(0) > 0.1U(2.5) + 0.90U(0)$$

i.e.,

$L_2 = (0, 0.11, 0.89) \succ L_2' = (0.1, 0, 0.9)$.

The latter is often violated.

Machina's paradox.

$C = \{$Trip to Venice, Watching an excellent movie about Venice, Staying home$\}$

Suppose decision maker prefers first to second to third outcome.

$L = \{99.9\%, 0.1\%, 0\}$

$L' = \{99.9\%, 0, 0.1\%\}$

$U(L) = 99.9\% U$(trip to Venice) $+ 0.1\% U$(watching a movie about Venice)

$> 99.9\% U$(trip to Venice) $+ 0.1\% U$(staying home)

$= U(L')$.

But you may be severely disappointed if you don't get to go to Venice and in that event, you may be better off if you stayed home - anticipating this you may choose L' over L.

Here, not being able to get to Venice may change your preference between the other two alternatives.

Regret.

Influence of what might have been on the utility obtained. Violates independence.

Money Lotteries and Risk Aversion.

Need lotteries with continuum of outcomes.

$C \subset \mathbb{R}$.

A monetary lottery is probability distribution on $\mathbb{R}$ summarized by a distribution function $F : \mathbb{R} \rightarrow [0, 1]$,

$$F(x) = \text{Pr}\{\text{Realized monetary payoff } \leq x\}.$$

If the distribution function is absolutely continuous, then it has a density function f associated with it and

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

Consider K lotteries $L_1, ..., L_K$ with respective distribution functions $F_1, ...F_K$ and the compound lottery $L = (L_1, ..., L_K; \alpha_1, ...\alpha_K)$. Then, the final distribution of money induced by this compound lottery is given by the distribution function F where:

$$\begin{aligned} F(x) &= \text{Pr}\{\text{lottery } L \text{ yields payoff} \leq x\} \\ &= \sum_{k=1}^{K} \text{Pr}\{L_k \text{ is realized}\} \, \text{Pr}\{L_k \text{ yields } \leq x\} \\ &= \sum_{k=1}^{K} \alpha_k F_k(x). \end{aligned}$$

We can just work with (simple) lotteries that are just distribution functions on the real line.

In particular, we choose

$\mathcal{L}$ = {set of all distribution functions over an interval $[a, \infty)$}

where a will be usually taken to be equal to zero.

Assume that there exists a utility function U with the expected utility form.

This implies that for each monetary outcome $x \in [a, \infty)$, there is a fixed real number $u(x)$ such that for any (lottery) distribution function F on $[a, \infty)$, we have

$$U(F) = \int_a^\infty u(x) dF(x)$$

which is the *expectation of the random variable* $u(x)$ when x follows the probability distribution F.

If in particular F is a discrete probability distribution assuming values $\{x_1, x_2, ...\}$ with probabilities $\{p_1, p_2, ...\}$, then

$$U(F) = \sum_i u(x_i) p_i$$

If F is an absolutely continuous probability distribution with density function f, then

$$U(F) = \int_a^{\infty} u(x)f(x)dx$$

The fixed real number $u(x)$ is interpreted as the utility of the deterministic (sure) monetary outcome x and $u : [a, \infty) \to \mathbb{R}$ is called the Bernoulli utility function

(as distinct from the von-Neumann Morgenstern utility function U which is defined on the space of lotteries $\mathcal{L}$ i.e., on probability distribution functions.)

Assume: Bernoulli utility function $u : [a, \infty) \to \mathbb{R}$ is increasing and continuous.