10.B.1

(a) Suppose an allocation is not weakly Pareto efficient. Then, there exists another allocation in which every individual is strictly better off. But in that case, it cannot be strongly Pareto efficient.

(b) In view of (a), it is sufficient to show that if an interior outcome is weakly Pareto efficient, then it is strongly Pareto efficient. Suppose, contrary to the statement, there is an interior outcome given by the allocation \((x_1, \ldots, x_I, y_1, \ldots, y_J)\) that is weakly Pareto efficient, but not strongly Pareto efficient. The latter implies there exists another feasible allocation \((x'_1, \ldots, x'_I, y'_1, \ldots, y'_J)\) where

\[
  u_i(x'_i) > u_i(x_i) \text{ for some } i
\]

and

\[
  u_k(x'_k) \geq u_k(x_k) \text{ for all } k = 1, \ldots, I.
\]

Since, \(x_i \in X_i = \mathbb{R}_{+}^L\) and preferences are strongly monotone,

\[
  u_i(x'_i) > u_i(x_i) \geq u_i(0)
\]

which immediately shows that \(x'_i \neq 0\) i.e., \(x'_{i,s} > 0\) for some commodity \(s\). As \(u_i\) is continuous and \(u_i(x'_i) > u_i(x_i)\), there exists \(\epsilon > 0\) small enough such

\[
  u_i(x'_{i1}, \ldots, x'_{i,s} - \epsilon, \ldots, x'_{iL}) > u_i(x_i)
\]

Now, construct a new allocation \((x''_1, \ldots, x''_I, y''_1, \ldots, y''_J)\) which same as \((x'_1, \ldots, x'_I, y'_1, \ldots, y'_J)\) except that \(\epsilon\) amount of commodity \(s\) is taken away from the consumption of consumer \(i\) and distributed equally among all other consumers. More formally,

\[
  y''_j = y'_j, j = 1, \ldots, J,
\]

and,

\[
  x''_{kl} = x'_{kl}, k = 1, \ldots, I, l \neq s,
\]

\[
  x''_{is} = x'_{is} - \epsilon
\]

\[
  x''_{ks} = x'_{ks} + \left(\frac{\epsilon}{I-1}\right), \text{ for all } k \neq i.
\]

We know that

\[
  u_i(x''_i) = u_i(x'_{i1}, \ldots, x'_{is} - \epsilon, \ldots, x'_{iL}) > u_i(x_i)
\]
Further, using strong monotonicity,

\[ u_k(x_k^*) > u_k(x_k) \text{ for all } k \neq i. \]

Thus, the allocation \((x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)\) makes all individuals better off compared to \((x_1, \ldots, x_I, y_1, \ldots, y_J)\). This contradicts weak Pareto efficiency of \((x_1, \ldots, x_I, y_1, \ldots, y_J)\).

[Note: we really don’t need interiority of the outcome for this result].

(c) Example: Consider a one good economy with two individuals and no production. The initial endowment of the good is 10. The utility of individual 1 is \(u_1(x_1) = 5\) (zero marginal utility for the good) and the utility of the other individual is \(u_2(x_2) = x_2\). The allocation where each consumer consumes 5 units of the good is clearly not strong Pareto efficient - as one can shift consumption from individual 1 to 2 and make the latter better off without making the former worse off (leaves him indifferent). But this allocation is weak Pareto efficient as no allocation can make both better off. Note the role played by strong monotonicity which is violated by individual 1’s utility function.

10.B.2. Since \(y_j^*\) solves

\[ \max_{y_j \in Y_j} p^* y_j \]

we have

\[ p^* y_j^* \geq p^* y_j, \forall y_j \in Y_j \]

and it follows (multiplying both sides of the inequality by scalar \(\alpha > 0\))

\[ \alpha p^* y_j^* \geq \alpha p^* y_j, \forall y_j \in Y_j \]

i.e., \(y_j^*\) solves

\[ \max_{y_j \in Y_j} (\alpha p^*) y_j. \]

Consumer i’s budget constraint at price \(\alpha p^*\)

\[ \alpha p^* x_i \leq \alpha p^* \omega_i + \sum_j \theta_{ij} (\alpha p^* y_j) \]

is exactly equivalent to

\[ p^* x_i \leq p^* \omega_i + \sum_j \theta_{ij} (p^* y_j) \]

the budget constraint at price \(p^*\). Therefore, the budget set of the consumers are unchanged between price vectors \(p^*\) and \(\alpha p^*\). Therefore, \(x_i^*\) solves consumer i’s utility maximization problem when the price vector is \(\alpha p^*\). Finally, since

\[ \sum_i x_{il}^* = \omega_l + \sum_j y_{jl}^*, l = 1, \ldots, L, \]

market clears at the given allocation and price \(\alpha p^*\).
10.C.2
(a) Consumer’s FOC:
\[ x(p) = \frac{\beta}{p} \]
(note consumer always consumes strictly positive quantity of good l; why?). Firm’s FOC:
\[ p = \sigma, \text{ if } q > 0 \]
\[ \leq \sigma, \text{ if } q = 0. \]

(b) \( p^* = \sigma, x^* = q^* = \frac{\beta}{\sigma}, m^* = \omega_m - \beta. \)

Firm’s supply curve (identical for all firms):
\[ q(p) = (\frac{p}{\beta})^{\frac{1}{\eta}} \]

Market supply:
\[ Q(p) = J(\frac{p}{\beta})^{\frac{1}{\eta}} \]

Market clearing:
\[ x(p) = Q(p) \]
i.e.,
\[ \alpha p^* = J(\frac{p}{\beta})^{\frac{1}{\eta}} \]
so that
\[ \ln \alpha + \epsilon \ln p = \ln J + \frac{1}{\eta}(\ln p - \ln \beta) \]
i.e.,
\[ \ln p = \frac{1}{1 - \epsilon \eta}(\ln \beta + \eta \ln \alpha - \eta \ln J) \]

Using firm’s supply curve:
\[ \ln q = \frac{1}{1 - \epsilon \eta}(\epsilon \ln \beta + \ln \alpha - \ln J). \]

To see the effects of change in parameters it is easier to look at elasticities such as
\[ \frac{\partial \ln p}{\partial \ln \alpha} = \frac{\eta}{1 - \epsilon \eta} \]
and so on. But you can also calculate the partial derivatives \( \frac{\partial m}{\partial \alpha} \) etc.

With one consumer and one firm, the maximization problem in equation (10.D.2) in the text, reduces to

$$\max_{x \geq 0} [\alpha + \beta \ln x - \sigma x + \omega_m]$$

which yields FOC (note that there must be an interior solution):

$$\bar{x} = \frac{\beta}{\sigma}$$

which is identical to the consumption and production level obtained in Problem 10.C..2.