Microeconomic Theory II
Spring, 2006.
Solution to PS1.

7.C.1. Player 1’s strategy set: (each strategy is a triplet specifying a move at each of the three information sets where she may be required to move):

\[
\{(L, x, x), (L, x, y), (L, y, x), (L, y, y), (M, x, x),
(M, x, y), (M, y, x), (M, y, y), (R, x, x), (R, x, y), (R, y, x), (R, y, y)\}
\]

Player 2’s strategy set: \{l, r\}.

7.E.1(a) Player 1’s strategy set: (each strategy is a triplet specifying a move at each of the three information sets where she may be required to move):

\[
\{(L, x, x), (L, x, y), (L, y, x), (L, y, y), (M, x, x), (M, x, y), (M, y, x), (M, y, y), (R, x, x), (R, x, y), (R, y, x), (R, y, y)\}
\]

8.B.1. In this problem, \(\alpha, \beta, w_i\) are nonnegative.

Payoff function of firm \(i\):

\[
\pi_i(h_1, ...h_I) = \alpha h_i + \alpha \sum_{j \neq i} h_j + \beta h_i (\prod_{j \neq i} h_j) - w_i^2(h_i).
\]

Therefore for any \(h_{-i}\), the unique best response of firm \(i\) is given by (take first order condition):

\[
\frac{1}{2w_i} [\alpha + \beta (\prod_{j \neq i} h_j)].
\]

If \(\beta = 0\), the strategy where firm \(i\) sets \(h_i = \frac{\alpha}{2w_i}\) is strictly dominant for firm \(i\) (as it is uniquely optimal for every \(h_{-i}\)). On the other hand, if \(\beta > 0\), the best response of firm \(i\) depends on \(h_{-i}\) so there is no strictly dominant strategy.

8.B.3. The strategy of each bidder is his bid \(b_i \geq 0\). The payoff of bidder \(i\) is \(v_i - p\), if he wins the auction and buys at price \(p\) (the second highest bid) and zero, if he doesn’t win the auction.

There are two possibilities for \(b_{-i}\):

(i) \(\max\{b_j : j \neq i\} \geq v_i\)

(ii) \(\max\{b_j : j \neq i\} < v_i\)

To show that it is weakly dominant strategy, it is sufficient to show that in all three cases, \(b_i = v_i\) yields at least as much payoff as any other strategy and that further, for every \(b'_i \neq v_i\), there is some \(b_{-i}\) for which \(b_i = v_i\) yields strictly higher payoff than \(b'_i\).

In case (i), \(b_i > v_i\) will lead to negative payoff if he wins auction and zero payoff otherwise while \(b_i < v_i\) will imply he will lose the auction and thus get zero payoff. In comparison, \(b_i = v_i\) always yields zero payoff in this case (where or not he wins the auction).
In case (ii), \( b_i = v_i \) implies \( i \) is the (only) winner and his payoff is \( v_i - \max\{b_j : j \neq i \} > 0 \). Any other bid \( b_i > \max\{b_j : j \neq i \} \) yields the same payoff. Any bid \( b_i < \max\{b_j : j \neq i \} \) makes him lose the auction and get zero payoff. The bid \( b_i = \max\{b_j : j \neq i \} \) yields expected payoff \( \frac{1}{n}[v_i - \max\{b_j : j \neq i \}] \) (where \( n \) is the number of bidders tied at the highest bid) which is \( < v_i - \max\{b_j : j \neq i \} \) as \( n \geq 2 \).

Finally, observe that any bid \( b'_i < v_i \) yields strictly lower payoff compared to \( b_i = v_i \) if \( b_{-i} \) is such that \( b'_i < \max\{b_j : j \neq i \} < v_i \). Also, any bid \( b'_i > v_i \) yields strictly lower payoff compared to \( b_i = v_i \) if \( b_{-i} \) is such that \( v_i < \max\{b_j : j \neq i \} \) \( < b'_i \).

Thus, \( b_i = v_i \) is a weakly dominant strategy. The last argument above also indicates why no other strategy \( b'_i \neq v_i \) can be weakly dominant (does not weakly dominate \( b_i = v_i \)). So, \( b_i = v_i \) is the unique weakly dominant strategy.

8.B.7.

Follows immediately from the fact that, for any given profile the expected utility from any mixed strategy cannot be strictly greater than the expected utility from every pure strategy in the strategy set of the player. More formally (when the strategy space is finite),

\[
\begin{align*}
\mathbb{E}(\sigma_i, \sigma_{-i}) &= \sum_{s_i \in S_i} \sigma_i(s_i)u_i(s_i, \sigma_{-i}) \\
&\leq \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}).
\end{align*}
\]


8.D.1. We know that \( a_4 \) and \( b_4 \) are not rationalizable. So they cannot be part of any mixed strategy NE (why? if the mixed strategy of player 2 puts positive probability mass on \( b_4 \) then he can do better by shifting the same mass to \( b_1 \) and \( b_3 \) equally; thus in a NE, mixed strategy of player 2 can put no probability mass on \( b_4 \); but then, player 1 must also be putting zero mass on \( a_4 \) as the latter can never be best response to any strategy of player 2 that puts zero prob mass on \( b_1 \).

Suppose there is a mixed strategy NE in which \( a_1 \) and \( a_3 \) are played with strictly positive probability. Then, both of these pure strategies must be yielding equal expected payoff to player 1. Check that this implies that player 2 plays \( b_1 \) and \( b_3 \) with equal probability - say, \( \alpha \); player 2 plays \( b_2 \) with probability \( 1 - 2\alpha \). Obviously, \( \alpha \in [0, \frac{1}{2}] \). The expected payoff to player 1 from playing \( a_1 \) or \( a_3 \) is then given by \( 7\alpha + 2(1 - 2\alpha) = 3\alpha + 2 \). However, the expected payoff to player 1 from playing pure strategy \( a_2 \) is then \( 10\alpha + 3(1 - 2\alpha) = 4\alpha + 3 > 3\alpha + 2 \). Therefore, there is no mixed strategy NE in which player 1 plays both \( a_1 \) and \( a_3 \) with strictly positive probability. This also rules out a mixed strategy NE where \( a_1, a_2 \) and \( a_3 \) are all played with strictly positive probability.

Suppose there is a mixed strategy NE in which \( a_1 \) and \( a_2 \) are the only strategies played with strictly positive probability. Then, player 2 would never play \( b_3 \) with strictly positive probability in his best response. Suppose player 2 plays \( b_1 \) and \( b_2 \) with probability \( \beta, 1 - \beta \) where \( \beta \in [0, 1] \). The expected payoff to player 1 from playing \( a_1 \) equals \( 2(1 - \beta) = 2 - 2\beta \) and playing \( a_2 \) yields...
\[ 5\beta + 3(1 - \beta) = 2\beta + 3 > 2 - 2\beta. \] Therefore, there is no mixed strategy NE in which player 1 plays both \( a_1 \) and \( a_2 \) with strictly positive probability.

Similarly, one can rule out a mixed strategy NE in which \( a_2 \) and \( a_3 \) are the only strategies played with strictly positive probability.

### 8.D.2.

Assuming the strategy set is finite, we know there exists a NE (possibly in mixed strategies) \( \sigma = (\sigma_1, \ldots, \sigma_I) \). Now suppose the strategy profile that survives iterated elimination of strictly dominated strategies is not a NE. Then, some strategy in the profile \( \sigma \) must have been eliminated as a dominated strategy. Let \( k \) be the first round of elimination in which some strategy in the profile \( \sigma \) was eliminated and let \( i \) be such that \( \sigma_i \) is eliminated in the \( k^{th} \) round of elimination. As this is the first iteration in which a strategy from \( \sigma = (\sigma_1, \ldots, \sigma_I) \) is eliminated, no strategy in \( \sigma_{-i} \) was eliminated by the \( k^{th} \) round. However, by the definition of NE:

\[ u_i(\sigma_i, \sigma_{-i}) \geq u_i(\tilde{\sigma}_i, \sigma_{-i}) \]

and this implies that \( \sigma_i \) could not have been a strictly dominated strategy in the \( k^{th} \) round of elimination. This leads to a contradiction.

### 8.D.3

This is a 2 players simultaneous move game where the strategy of each player \( i \) is his bid \( b_i \geq 0 \) and the payoff function is

\[ u_i(b_i, b_j) = \begin{cases} 0, & \text{if } b_i < b_j, \\ \frac{1}{2}(v_i - b_i), & \text{if } b_i = b_j, \\ (v_i - b_i), & \text{if } b_i > b_j. \end{cases} \]

(a) Suppose that for player \( i \), there is a bid \( b_i \) that is strictly dominated by some other bid \( b'_i \). If player \( j \) bids \( b_j = \max\{b_i, b'_i\} + 1 \), then \( u_i(b_i, b_j) = u_i(b'_i, b_j) = 0 \), a contradiction.

(b) It is easy to check that the bid \( b_i = v_i \) weakly dominates every bid \( b'_i > v_i \). To see this observe that

(i) if \( b'_i < b_j \), player \( j \) wins the auction and so both \( b_i, b'_i \) yield zero payoff

(ii) if \( b_j \in (v_i, b'_i] \), player \( i \) wins the auction (with at least 0.5 probability) when he bids \( b'_i \) and obtains negative payoff (bid exceeds valuation) whereas he could earn zero payoff (does not win auction) by bidding \( b_i \)

(iii) if \( b_j = v_i \), player \( j \) wins the auction with probability one when he bids \( b'_i \) and thus gets negative payoff whereas he wins auction with probability \( \frac{1}{2} \) when he bids \( b_i \) and thus gets zero expected payoff (bid equals valuation).

(iv) if \( b_j < v_i \), then player \( j \) wins auction with both bids, bid \( b_i \) yields zero payoff while \( b'_i \) yields negative payoff. Also, check that if \( v_i > 2 \), \( b_i = 1 \) weakly dominates \( b'_i = 0 \).

Further, check that if \( v_i \geq 1 \), \( b_i = v_i - 1 \) weakly dominates \( b'_i = v_i \).

(You need to work out the details).

### 8.D.5
Let $x_i \in [0, 1]$ denote the location of player $i$.

(a) The payoff function is as follows (why?):

If $0 \leq x_i < x_j \leq 1$, then

$$u_i(x_i, x_j) = \frac{x_i + x_j}{2}$$
$$u_j(x_i, x_j) = 1 - \frac{x_i + x_j}{2}.$$ 

If $0 \leq x_i = x_j \leq 1$, then

$$u_i(x_i, x_j) = \frac{1}{2} = u_j(x_i, x_j).$$

It is easy to show that $x_1 = x_2 = \frac{1}{2}$ is a NE. Any firm that deviates (moves slightly to the right or left) will obtain payoff (market share) less than $\frac{1}{2}$.

To show that this is the unique NE, note that any situation where $x_1 = x_2 \neq \frac{1}{2}$ cannot be a NE (for example, if $x_1 = x_2 < \frac{1}{2}$, then both firms earn payoff equal to $\frac{1}{2}$ and one firm can do better by moving slightly to the right). Also, $x_1 \neq x_2$ cannot be a NE, because the firm to the right can do better by moving slightly to the left.

(b) Suppose $(x_1, x_2, x_3)$ is a pure strategy NE. If $x_1 = x_2 = x_3$, then each firm earns payoff $\frac{1}{4}$, in which case one firm can do better by moving to the right if $x_1 = x_2 = x_3 < \frac{1}{3}$ or to the left if $x_1 = x_2 = x_3 > \frac{1}{3}$. If $x_i = x_j < x_k$, then firm $k$ can do better by moving slightly to the left. If $x_i = x_j > x_k$, then firm $k$ can do better by moving slightly to the right. If $x_i < x_j < x_k$, then firm $k$ can do better by moving slightly to the left. This rules out all possibilities.

8.D.9

(a) No standard answer. In view of multiplicity of NE, most people feel that in the absence of preplay communication, coordination may break down and so playing LL, L or R is quite risky as one does not know what player 1 will be playing - M seems reasonable to such people.

(b) The two pure strategy NE are (U, LL) and (D,R).

There is only one mixed strategy NE. Player 2 mixes between LL and L with probability $\frac{1}{2}$ each while player 1 mixes between U and D with probability $\frac{21}{52}$ and $\frac{31}{52}$. Check that given these probabilities, player 1 is indifferent between U and D (both yield payoff of 0) and player 2 is indifferent between LL and L (both yield payoff of $\frac{1}{2}$); further, given the strategy of player 1, player 2 obtains 0 from playing M and negative payoff from playing R. Thus, this satisfies all the conditions for being a mixed strategy NE.

Check that no other mixed strategy NE exists. For example, if player 2 mixes over $\{LL, M\}$, then he can be indifferent between them only if player 1 plays U with prob. $\frac{21}{52}$ and D with prob. $\frac{31}{52}$. But in that case player 2 gets zero expected payoff from both LL and M; however her expected payoff from L is strictly positive and this violates the condition for being a mixed strategy NE. And so on...
(c) If M was your choice in part (a), then you can see M is not part of any NE. M is however rationalizable as it is the unique best response of player 2 if player 1 plays U and D with probability $\frac{1}{2}$ each.

(d) Preplay communication may resolve coordination problem and makes it more likely that one of pure strategy NE may be played (they are also Pareto efficient).

8.E.1.

The strategy of each player specifies an action contingent on type. There are four pure strategies for each player:

- AA: Attack if strong, Attack if weak
- AN: Attack if strong, Not attack if weak
- NA: Not attack if strong, Attack if weak
- NN: Not attack if strong, Not attack if weak

The expected payoffs to each player from each pair of strategies can be computed and represented as a matrix.

$$
\begin{bmatrix}
AA & AN & NA & NN \\
\frac{M}{4} - s + \frac{w}{4} & \frac{M}{2} - s & \frac{3M}{4} - \frac{s + w}{4} & \frac{M}{4} - \frac{s + w}{4} \\
\frac{M}{4} - s & \frac{M}{2} - s + \frac{w}{4} & \frac{M}{4} - s & \frac{3M}{4} - \frac{s + w}{4} \\
\frac{M}{4} & \frac{M}{2} & \frac{M}{4} - s & \frac{3M}{4} - \frac{s + w}{4} \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

Any NE of this normal form game is a Bayesian NE of the original game. There are various pure strategy NE under various restrictions on the parameters $M, s, w$. We know that $w > s \geq 0, M \geq 0$.

The way to proceed is to consider each of the 16 combinations of strategies and see whether and under what conditions on the parameters can such a combination be a NE of the bimatrix game. For example, $(AA, AA)$ can never be a BNE because if player 2 plays AA, player 1 can always do better by deviating and playing AN (since $w > 0$). Similarly, $(NN, NN)$ can never be a BNE.

$(AA, AN),$ $(AN, AA),$ $(AA, NN),$ $(NN, AA),$ $(AN, AN)$ can all be sustained as BNE under certain conditions on the parameters.

For example, $(AA, AN)$ is a BNE if

(i) $AA$ is a best response by player 1 to AN in the bimatrix game which requires:

$$
\frac{M}{2} - s - s + \frac{w}{4} \geq \frac{M}{4} - \frac{s + w}{4}
$$

since $\frac{M}{2} - s + \frac{w}{4} \geq 0$

(ii) $AN$ is a best response by player 2 to AA which requires

$$
\frac{M}{4} - s \geq 0
$$

and the above inequalities can be jointly satisfied iff

$$
M \geq \max\{w, 2s\}.
$$

The same condition also ensures that $(AN, AA)$ is a BNE.
One can work out similarly...

8.E.3

Define a firm to be of type $H$ if its unit production cost is $c_H$ and of type $L$ if its unit production cost is $c_L$.

Let $q^\tau_i$ denote the quantity produced by firm $i$ of type $\tau$, $i = 1, 2$, $\tau = H, L$.

The expected profit of firm $i$ of type $\tau$ is given by:

$$(1 - \mu)[(a - b(q^H_i + q^H_j) - c_\tau)q^\tau_i] + \mu[(a - b(q^L_i + q^L_j) - c_\tau)q^\tau_i]$$

and maximizing this with respect to $q^\tau_i$ (given any strategy $(q^H_j, q^L_j)$ of firm $j, j \neq i$) yields the following first order condition for an interior solution:

$$(1 - \mu)(a - 2bq^\tau_i - q^H_j - c_\tau) + \mu(a - 2bq^\tau_i - q^L_j - c_\tau) = 0$$

so that

$$q^\tau_i = \frac{a - c_\tau + [(1 - \mu)q^H_j + \mu q^L_j]}{2b}$$

This is the best response of firm $i$ of type $\tau$ (with the understanding that the best response is zero if the right hand side is negative). Solving four equations ($i = 1, 2, j \neq i, \tau = H, L$) yields the Bayesian Nash Equilibrium:

$$q^H_1 = q^H_2 = \frac{a - c_H + \frac{\mu}{2}(c_L - c_H)}{3b}$$

$$q^L_1 = q^L_2 = \frac{a - c_L + \frac{\mu}{2}(c_H - c_L)}{3b}$$

One trick to ease computation is to recognize the symmetry of the model and conjecture a solution where $q^H_1 = q^H_2 = q^H, q^L_1 = q^L_2 = q^L$ so that one only solves two equations in two unknowns.

Other problems:

1. Player 1 plays T and B with probability $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Player 2 plays L and R with probability $\frac{3}{4}$ and $\frac{1}{4}$, respectively.

2. If a pure strategy $s_i$ that is played with strictly positive probability $\mu > 0$ in a mixed strategy NE is strictly dominated by some other strategy $\sigma_i$ (possibly mixed), then player $i$ can deviate and improve his payoff by playing a new mixed strategy where he plays $s_i$ with probability $0$ and instead shifts the probability mass $\mu$ to the strategy $\sigma_i$ (leaving other things unchanged). This violates the definition of a NE.

Suppose there is a mixed strategy NE $\sigma = (\sigma_1, ..., \sigma_I)$ where at least one one of the pure strategies played with strictly positive probability by some player is eliminated during iterated deletion of strictly dominated strategies. Let $s_i$ be the first of such strategies to be deleted. This implies that at that round of iterated elimination, all of the strategies in $\sigma_{-i}$ have not been eliminated. Since $s_i$ is strictly dominated in that round

$$u_i(s_i, \sigma_{-i}) < u_i(\sigma'_i, \sigma_{-i})$$

for some $\sigma'_i \in \Delta(S_i)$. But this violates the definition of NE as $u_i(s_i, \sigma_{-i})$ is in fact the payoff to player $i$ in the supposed NE.