Dynamic Sorting in Durable Goods Markets with Buyer Heterogeneity*

Santanu Roy†
Southern Methodist University, Dallas, Texas.

November 13, 2012

Abstract

In a competitive durable good market where sellers have private information about quality, I identify certain inefficiencies that arise due to heterogeneity in buyers’ valuations. Even if the dynamic price mechanism induces higher quality sellers to sell later and all goods are eventually traded, inefficiency can arise because high valuation buyers may buy early when low quality goods are sold, while high quality goods are allocated to low valuation buyers that buy later. This misallocation adds to inefficiency caused by delay in trading. Under certain circumstances, high quality goods may never be traded as in the classical static "lemons" outcome.

Key-words: Asymmetric Information, Adverse Selection, Durable Good, Dynamic Trading, Dynamic Sorting, Heterogenous Buyers.

JEL Classification: D82, L15.

*I thank Alessandro Lizzeri and Maarten Janssen for their comments on an earlier version of this paper.
†Department of Economics, Southern Methodist University, Dallas, TX 75275-0496. Tel: 214 768 2714. E-mail: sroy@smu.edu.
1 Introduction

It is well known that when sellers have private information about the quality of their goods and potential buyers cannot observe quality prior to purchase, market outcomes can be characterized by inefficiency due to adverse selection. In static markets, this "lemons problem" (Akerlof, 1970, Wilson, 1980) manifests itself in insufficient trading; better quality goods may not be traded with consequent loss of gains from trade. However, most of the standard examples of markets with adverse selection involve durable goods. Over the last decade, a growing literature has studied the manner in which the dynamic nature of such markets may endogenously resolve some of the trading problems predicted by static models. This paper is a contribution to a strand of this literature that has emphasized the role of time (and waiting to trade) as a sorting device in such markets (even when there are no other possibilities of signaling or screening of private information).\(^1\)

Janssen and Roy (2002) analyze the dynamic competitive price mechanism in a durable good market where buyers are homogenous and exit the market after trading, and no entry of traders occurs after the initial period. They show that all goods are traded eventually, with better quality goods being traded later and at higher prices. The owner of a lower quality good has lower incentive to wait; further, once lower quality goods are traded, buyers’ expectation of quality and willingness to pay increase, which allows prices to increase over time.\(^2\) The main source of inefficiency in the market is the cost of waiting, and the extent of waiting on the equilibrium path is related to impatience.\(^3\) This result has been extended to models with entry of cohorts of sellers over time (Janssen and Karamychev, 2002, Janssen and Roy, 2004). Similar qualitative results have been derived in markets with search frictions (Inderst and Müller 2002), decentralized trading with frictions (Blouin 2003, House and Leahy 2004, Camargo and Lester 2011, Kim 2011) and models of dynamic auctions (Vincent 1990).\(^4\)

\(^1\)A different strand of the literature on durable goods markets with adverse selection has examined the role of interlinked primary and secondary markets for distinct vintages and endogenous sorting of traders in these markets as a way resolving the classical lemons problem. Heterogeneity in preferences of agents can play an important role in this process (see, among others, Hendel, Lizzeri and Siniscalchi 2004).

\(^2\)If buyers can acquire information about quality through inspection, then time on the market can have somewhat different informational content. See, Taylor (1999).

\(^3\)In a recent paper, Fuchs and Skrypacz (2012) argue that efficiency may be improved by closing down trading for certain lengths of time (rather than allowing continuous trading that leads to smooth screening of seller types).

\(^4\)Somewhat similar dynamics may be generated when there is an exogenous public information generating process that reveals the private information of sellers over time, see Daley and Green (2012). See also,
The purpose of this paper is identify certain problems that can emerge in such markets when we allow for heterogeneity of buyers. I study a simplified version of the model in Janssen and Roy (2002) with two quality types (high and low) and two types of buyers (high and low valuations). High valuation buyers are willing to pay higher quality premium and there are enough high valuation buyers to potentially buy all high quality goods; the first best (full information) allocation is one where all goods are traded and high quality goods are bought only by high valuation buyers. The problem with implementing this allocation through the dynamic market mechanism is not just that the latter requires inefficient delay in trading (as recognized in the existing literature), but that high valuation buyers may not want to wait to buy high quality goods in later periods. While such buyers gain more (than low valuation buyers) from consuming high rather than low quality, the utility from current consumption that is lost (due to discounting) is also higher for a high valuation buyer. When the latter effect is strong, the equilibrium path may be one where though all goods are traded (maybe even within the first two time periods), the allocation of quality types across buyers is distorted - low valuation buyers buy high quality and high valuation buyers buy low quality. This allocational inefficiency is directly linked to the dynamic nature of the market and the impatience to consume; it cannot occur in a static market. This misallocation should be viewed as another manifestation of the dynamic lemons problem. Further, when only low valuation buyers remain in the market in later periods, they may not be willing to pay a price higher than the reservation price of the high quality sellers; in that case, high quality goods may not be traded at all and the outcome is similar to the static "lemons" problem.

The next section outlines the framework and some basic properties of the equilibrium. Section 3 discusses the outcome in a version of our model with homogenous buyers. Section 4 contains the main results of the paper on inefficient allocation of quality across buyer types. Section 5 contains results on impossibility of trading high quality goods under certain conditions. Section 6 concludes.

2 Framework

Consider a dynamic competitive market for a perfectly durable good. Time is discrete and denoted by $t = 1, 2, \ldots, \infty$. There is a continuum of price taking sellers and buyers that are a recent Somewhat different outcomes may emerge in a sequential bargaining framework; see Hörner and Vieille (2009).
ininitely lived. All traders enter the market in the initial period, and no entry occurs after period 1. Traders leave the market immediately after trading. All traders are risk neutral and discount the future using a common discount factor $\delta \in (0, 1)$. The valuation of a buyer or a seller for a unit of the good of a certain quality reflects the discounted sum of flow utility from owning it forever.\footnote{An alternative interpretation is that each seller has the ability (or the resource) to produce one unit of the good at a cost equal to its valuation in a time period of her choice.}

The total mass of sellers is 1. Each seller is endowed with one unit of the good. A seller’s valuation of the good she is endowed with depends on quality and without loss of generality, is assumed to be identical to quality. The quality of the good is either high ($\theta_H$) or low ($\theta_L$) where $0 < \theta_L < \theta_H$. A fraction $\alpha \in (0, 1)$ of sellers own high quality goods and the rest own low quality; thus, there are $\alpha$ high quality goods and $(1 - \alpha)$ low quality goods to be traded in this market. Each seller knows the quality of her own good, but quality is not observed by buyers prior to purchase. Note that the flow utility per period to a seller owning quality $\theta_s$ is $(1 - \delta)\theta_s$. The discounted net surplus (evaluated in the initial time period) of a seller who owns a unit of and sells it at price $p_t$ in period $t$ is

$$\left\{ \left( \sum_{i=1}^{t-1} \delta^{i-1} \right) ((1 - \delta)\theta_s) + \delta^{t-1} p_t \right\}, \text{ if } t \geq 2$$

$$p_1, \text{ if } t = 1$$

(1)

Observe that for a seller with quality $\theta_s$, the difference in net surplus or the incentive to not sell in period $t$ and instead wait to trade in period $t + k, k \geq 1$, is given by:

$$[\delta^k p_{t+k} - p_t] + \left( \sum_{i=1}^{k-1} \delta^{i-1} \right) ((1 - \delta)\theta_s)$$

which is strictly increasing in $\theta_s$ so that other things being equal, a high quality seller has greater incentive to wait to trade than a low quality seller.

The total mass of buyers is denoted by $\mu$; we assume that $\mu > 1$. Buyers have unit demand and are risk neutral. There are two types of buyers: high valuation (type $H$) and low valuation (type $L$). Let (upper case) $V_H^H$ and $V_L^L$ denote respectively the valuations of the high and low quality good by a high valuation buyer. For a low valuation buyer, the valuations of high and low quality goods are denoted by (lower case) $v_H^H$ and $v_L^L$ respectively. We assume that:
Thus, independent of her type, every buyer’s valuation of the high quality good exceeds that of the low quality good. Further, for each quality, high valuation buyers are willing to pay more than low valuation buyers. A high valuation buyer’s willingness to pay for either quality good exceeds the seller’s valuation of that quality. A low valuation buyer’s willingness to pay for low quality exceeds the seller’s valuation of low quality, but this need not necessarily be true for a high quality good.

The measure of high valuation buyers is denoted by $\beta$. Recall that $\alpha$ is the measure of high quality sellers. We will assume that:

$$0 < \alpha < \beta < 1 - \alpha < 1$$

i.e., there are more of low quality goods than high quality goods and further, there are sufficient high valuation buyers to buy all high quality goods, but not enough of them to buy all low quality goods. This implies that in order for all low quality goods to be traded, some low valuation buyers must also buy. In equilibrium, all low quality goods are necessarily sold and therefore, some low valuation buyers always buy.

Though buyers do not directly observe the quality of goods offered for sale, they anticipate (correctly, in equilibrium) the proportion of high and low quality goods offered for sale in every period. For a high valuation buyer who anticipates that fraction $\pi_t \in [0, 1]$ of goods offered for sale in period $t$ is of high quality, the discounted net expected surplus from buying in period $t$ at price $p_t$ is given by:

$$\delta^t \times \left[ \pi_t V^H + (1 - \pi_t) V^L \right] - p_t$$

The expected net discounted surplus of a low valuation buyer is similar. Not buying at all yields a net surplus of zero (for either type of buyer).

We assume that:

$$V^H - V^L > \max\{v^H, \theta_H\} - v^L.$$  \hspace{1cm} (5)

(5) implies that high valuation buyers are willing to pay higher quality premium than low valuation buyers; further, even if there is no gain from trading a high quality good with a low valuation seller, trading the good with a high valuation buyer yields more total
surplus than not trading it at all and instead letting the high valuation buyer consume a low quality good. As a result, it is socially optimal to trade all goods and to allocate all high quality goods to high valuation buyers. Indeed, using (5), it is easy to check that under full information, there is an equilibrium where all goods are traded in the initial period and further, all high quality goods are bought by high valuation buyers and all high valuation buyers buy; this is also the first best allocation.

In order to ensure that there is a "lemons problem" in the one-period version of the model we assume:

\[ \alpha V^H + (1 - \alpha) V^L < \theta_H \]  

(6)

This implies that in a static market, only low quality goods are traded.

*All of the assumptions outlined above are assumed to hold throughout the paper.*

To understand the difference in incentive to wait of the two types of buyers, suppose that only low quality goods are sold in period 1 and only high quality goods in period 2. For a high valuation buyer, the incentive to wait and buy in period 2 i.e., the difference in net surplus, is given by:

\[ \left[ \delta V^H - V^L \right] - \left[ \delta p_2 - p_1 \right] = \delta \left[ V^H - V^L \right] - (1 - \delta) V^L - \left[ \delta p_2 - p_1 \right] \]

while that for low valuation buyers is given by:

\[ \left[ \delta v^H - v^L \right] - \left[ \delta p_2 - p_1 \right] = \delta \left[ v^H - v^L \right] - (1 - \delta) v^L - \left[ \delta p_2 - p_1 \right]. \]

In the expressions on the right hand side, the first term in square brackets reflects the quality premium that each type of buyer is willing to pay and using assumption (5), this is greater for high valuation buyers which tends to make such buyers more willing to wait. However, the second term, which has a negative sign, reflects the foregone flow utility of owning a low quality good for one period; as high valuation buyers derive higher utility from consumption of both qualities, this term tends to make high valuation buyers less willing to wait. When the second term dominates, low valuation buyers are the ones that

\[ \text{For example, in period 1, all low quality goods can be sold at price } v^L (\theta_L) \text{ and all high quality goods at price } [V^H - (V^L - v^L)] (> \theta_H), \text{ using (5)). } \]

\[ \text{H-type buyers are indifferent between buying high and low quality goods and all H-type buyers buy. L-type buyers are indifferent between buying low quality goods and not buying at all, and using (5), they earn negative surplus if they buy the high quality good (at that price).} \]
wait to buy later.

We now analyze the dynamic equilibrium outcomes under incomplete information.

**Definition 1** A dynamic equilibrium is defined by a sequence of prices $p_t \geq 0$, anticipated proportions $\pi_t \in [0, 1]$ of high quality among goods offered for sale in each period $t$, $t = 1, 2, 3, \ldots$, a purchase period for each buyer (may be infinite, which implies "never buy") and a selling period for each seller (may be infinite, which means "never sell") such that: (i) Buyers and sellers maximize their discounted expected net surplus; (ii) The market clears every period; (iii) if $t > 1$ and the total measure of low quality goods traded in periods prior to period $t$ is $(1 - \alpha)$ i.e., all low quality goods have been traded in the past, then $\pi_t = 1$ (iv) If strictly positive measure of trade occurs in any period, then the actual proportion of high and low quality goods traded in that period equals the anticipated proportions for that period.

Condition (iii) is a mild restriction on demand in periods of no trade; if (almost) all low quality goods have been sold in the past, buyers should expect quality to be high with probability one.

The following useful properties of dynamic equilibrium are easy to check:

**R.1:** A low quality seller never sells later than a high quality seller.

**R.2:** There can be at most one period in which both high and low quality goods are traded.

**R.3:** No high quality good is sold in the first period of trading.

**R.4:** All low quality goods are traded.

**R.5:** In every period $t$, the equilibrium price satisfies

$$p_t \geq v_L$$  \hspace{1cm} (7)

**R.6:** There can be at most one period in which only low quality goods are traded, and at most one period in which only high quality goods are traded.

**R.1** follows from the fact that if a high quality seller sells in period $\tau$, then

$$0 \leq [p_\tau - \theta_H] \geq \delta^{k-1}[p_{\tau+k} - \theta_H], \forall k \geq 1$$

so that

$$0 < [p_\tau - \theta_L] > \delta^{k-1}[p_{\tau+k} - \theta_L], \forall k \geq 1$$
which implies that all low quality sellers sell in some period \( t \leq \tau \). \( R.2 \) follows immediately from \( R.1 \). To see \( R.3 \), observe that if high quality sellers find it optimal to sell in the first period in which trade occurs, then all low quality sellers would strictly prefer to sell in that period, and this leads to a contradiction (using (6)) as even the high valuation buyers’ willingness to pay for the expected quality sold would fall short of the high quality seller’s reservation price. \( R.4 \) follows from the fact that if some low quality goods are never traded, then \( p_t \leq \theta_L, \forall t \geq 1 \), but since \( v^L > \theta_L \) all buyers would strictly prefer to buy in some period, a contradiction (as \( \mu > 1 \), there must be excess demand in some period). \( R.5 \) holds as \( p_t < v^L \) in any period implies that all buyers earn strictly positive surplus in equilibrium (valuation is at least as large as \( v^L \) in every time period) and as there are more buyers than sellers, there must be excess demand in some period. Finally, to see \( R.6 \), note that if there are two periods \( t, t' \) in which only low quality goods are traded and, without loss of generality, \( t < t' \), then for a low quality seller to be indifferent between both periods we need \( p_t < p_{t'} \); however, in that case a buyer buying in period \( t' \) would strictly gain by buying in period \( t \) instead. The same argument holds for high quality.

3 Digression: Case of Homogenous Buyers

Janssen and Roy (2002) show that when all buyers are homogenous and the distribution of quality is continuous on an interval, then all goods are traded in finite time. It is easy to check that even with a discrete distribution of quality as in this paper, the result continues to hold. However, for high \( \delta \), trading necessarily involves intermediate periods of no trade in order to prevent waiting by low quality sellers.

**Proposition 1** (Janssen and Roy, 2002). Suppose that all buyers are homogenous and in particular,

\[
V^H = v^H = \bar{V} > \theta_H, V^L = v^L = \bar{V} > \theta_L, \bar{V} > \bar{V}.
\]

Then, there exists a dynamic equilibrium where all goods are traded in finite time. Further, all goods are traded in finite time in every dynamic equilibrium.

**Proof.** If \( \bar{V} - \theta_L \geq \delta [\bar{V} - \theta_L] \), then all low (high) quality goods are traded in period 1 (period 2) at price \( \bar{V} (\bar{V}) \). If \( \bar{V} - \theta_L < \delta [\bar{V} - \theta_L] \), let \( \tau \geq 1 \) be the smallest positive integer such that

\[
\bar{V} - \theta_L > \delta^\tau [\bar{V} - \theta_L].
\]
Then, for $\epsilon > 0$ sufficiently small, there is an equilibrium where $(1 - \alpha - \epsilon)$ low quality goods are sold in period 1 at price $V$, while $\epsilon$ low quality goods and $\alpha$ high quality goods are sold in period $1 + \tau$ at a price equal to the buyers’ expected valuation in that period. No trade occurs in periods $2, \ldots, \tau$ (set $p_t = 0, \pi_t = V$ in those periods; this does not violate part (iii) of the definition of equilibrium as there are unsold low quality goods in the market).

To see the second part of the proposition, suppose there is a dynamic equilibrium where some high quality goods are not traded. This implies

$$p_t \leq \theta_H, \forall t. \tag{9}$$

It is easy to check that $R.1 - R.6$ continue to hold under the hypothesis of this proposition. In particular, all low quality goods must be traded. There are two possibilities: (a) only low quality goods are traded; (b) some (but not all) high quality goods are traded. In case (a) as $\mu > 1$, $p_t = V$ in all periods where trade occurs; as low quality sellers would then strictly prefer to sell in the first of these periods, there can be only one period in which trade occurs, denote this period by $\tau$. In case (b), let $\tau$ be the first period in which high quality goods are sold and using $R.1, R.2, \tau > 1$ and all low quality goods are sold on or before period $\tau$. In both cases, only high quality goods are left unsold after period $\tau$. Using condition (iii) of the definition of dynamic equilibrium, $\pi_{\tau+1} = 1$ and as $V > \theta_H$, (9) leads to excess demand in period $\tau + 1$, a contradiction. $\blacksquare$

In the dynamic equilibrium constructed in the proof of Proposition 1, the length of time before which high quality good can be traded is increasing in the discount factor. In particular, while all goods are traded within two periods if the discount factor $\delta$ lies below a critical level $\frac{V - \theta_H}{V - \theta_L}$, for higher values of the discount factor, the length of time before which all goods can be traded becomes larger as the discount factor becomes larger (this can be seen by looking at the effect of change in $\delta$ on $\tau$ in (8)). Even though dynamic trading allows high quality goods to be traded, it is when agents are extremely impatient, even myopic, that the delay in trading is likely to be small. Low discounting tends to create more delay and inefficiency in trading. Indeed, the delay in trading high quality becomes infinitely large as agents become perfectly patient i.e., $\delta \rightarrow 1.7$ Usually the outcome of a dynamic model approaches the outcome of a static version of the model as $\delta \rightarrow 0$. In sharp contrast, in the framework of Proposition 1, it is when $\delta \rightarrow 1$ that the outcome of the dynamic model approaches the outcome of the static version of the model.

7This can be shown to be true for all equilibria.
In the static version of the model, the lemons problem is manifest in the fact that high quality goods are not traded and inefficiency arises due to associated loss in gains from trade. As the above proposition indicates, in the dynamic (durable good) version of the market for lemons with homogenous buyers, all goods are eventually traded and the gains from trade are eventually realized. The "lemons problem" manifests itself in a different form: traders need to wait before trading high quality. Dynamic inefficiency is related to the cost of waiting (which depends, among other things, on the extent of discounting).

The main contribution of this paper, described in the next section, is to show that when one allows for heterogeneity of buyers, delay in trading is not the only kind of inefficiency; there may be an additional inefficiency related to the intertemporal allocation of qualities across various buyer types.

4 Heterogenous Buyers: Inefficient Allocation

Recall that in the first best allocation (full information market outcome), all high quality goods are bought by high valuation buyers. We now show that under certain conditions, the dynamic equilibrium allocation is necessarily one where high quality goods (that are traded) are actually bought by low valuation buyers. Obviously, this requires us to assume that low valuation buyers are willing to pay the reservation price of sellers for high quality goods:

(A1) \( v^H > \theta_H \).

Assumption (A1) is retained throughout this section.

Proposition 2 Assume (A1). Suppose that either,

\[
\delta < \min \left\{ \frac{v^L - \theta_L}{V^H - \theta_H}, \frac{v^L - v^L}{V^H - v^H} \right\}
\]

or,

\[
\frac{v^L - v^L}{V^H - v^H} > \frac{v^L - \theta_L}{V^H - \theta_H} \quad \text{and} \quad \frac{v^L - \theta_L}{V^H - \theta_H} > \frac{v^L - \theta_L}{\theta_H - \theta_L}.
\]

Then there does not exist a dynamic equilibrium where high quality goods are traded and all traded high quality goods are bought by high valuation buyers.

Proof. Suppose that contrary to the proposition, there exists a dynamic equilibrium where a strictly positive measure of high quality goods is (eventually) sold and all high quality
goods that are sold are bought by high valuation buyers. Let $T$ be the first period in which high quality goods are sold. Then, from $R.3$, $R.4$ and $R.6$, we have that $T > 1$, all low quality goods are sold, some low quality goods must be sold before period $T$, and only high quality goods are sold after period $T$. No low valuation buyers buy on or after period $T$ (for otherwise some high quality goods would go to low valuation buyers). Let $t$ be the last period before $T$ in which trade occurs (there must be such a period in order for high quality goods to be sold in period $T$); then only low quality goods are sold in period $t$. Using $R.6$, no trade occurs before period $t$. As there are not enough high valuation buyers to buy all low quality goods, some low valuation buyers must buy, and as they buy only in period $t$ we have $p_t \leq v^L$. Using $R.5$, $p_t \geq v^L$. Thus,

$$p_t = v^L. \quad (12)$$

There are two possibilities: (i) Both high and low quality goods are sold in period $T$ $(0 < \pi_T < 1)$; (ii) only high quality goods are sold in period $T$ $(\pi_T = 1)$. Consider case (i). Low quality sellers are indifferent between selling in periods $t$ and $T$:

$$p_t - \theta_L = \delta^{T-t}[p_T - \theta_L]$$

so that using (12), we have

$$\delta^{T-t}p_T = v^L - (1 - \delta^{T-t})\theta_L \quad (13)$$

On the other hand, $H$-type buyers buy in period $T$ so that:

$$V^L - p_t \leq \delta^{T-t}\left\{\pi_T V^H + (1 - \pi_T) V^L\right\} - p_T$$

$$\leq \delta^{T-t}[V^H - p_T]$$

so that using (12), we have

$$\delta^{T-t}p_T \leq \delta^{T-t}V^H - (V^L - v^L) \quad (14)$$

From (13) & (14) we have:

$$\delta^{T-t} \geq \frac{V^L - \theta_L}{V^H - \theta_L} \quad (15)$$
As high quality sellers sell in period $T$,

$$p_T \geq \theta_H.$$  \hfill (16)

Using (13) and (16) we have:

$$\delta^{T-t} \leq \frac{v^L - \theta_L}{\theta_H - \theta_L}.$$  \hfill (17)

Combining (15) and (17):

$$\frac{v^L - \theta_L}{\theta_H - \theta_L} \leq \delta^{T-t} \leq \frac{v^L - \theta_L}{\theta_H - \theta_L}.$$  \hfill (18)

If (10) holds, then $\delta < \frac{v^L - \theta_L}{\theta_H - \theta_L}$ which violates (18). If (11) holds, then $\frac{v^L - \theta_L}{\theta_H - \theta_L} > \frac{v^L - \theta_L}{\theta_H - \theta_L}$ which violates (18). This eliminates case (i).

Next, consider case (ii). In this case using R.6, all low quality goods are sold in some period $t < T$, and all high quality goods are sold in period $T$. Suppose $T > t + 1$. As all goods are sold to high valuation buyers in period $T$, all low quality goods are sold in period $t$ and all buyers expect high quality with probability one after period $t$, there is no trade in period $t + 1$ only if $p_{t+1} \geq V^H$; but in that case, all high quality sellers would strictly prefer to sell in period $t + 1$ rather than wait for a later period because the highest price they can ever get in any period with trading is $V^H$. Therefore, $T = t + 1$. Since low quality sellers prefer to sell in period $t$:

$$p_t - \theta_L \geq \delta[p_{t+1} - \theta_L]$$

and using (12) we have

$$\delta p_{t+1} \leq v^L - (1 - \delta)\theta_L.$$  \hfill (19)

Since L-type buyers buy in period $t$, using (12),

$$0 = v^L - p_t \geq \delta[v^H - p_{t+1}]$$

so that

$$p_{t+1} \geq v^H.$$  \hfill (20)

From (19) and (20):

$$\delta \leq \frac{v^L - \theta_L}{v^H - \theta_L}.$$  \hfill (21)
H-type buyers buy in period \( t + 1 \) and therefore:

\[
V^L - p_t \leq \delta [V^H - p_{t+1}]
\]

so that (using (12)):

\[
\delta p_{t+1} \leq \delta V^H - (V^L - v^L) \tag{22}
\]

From (20) and (22)

\[
\delta \geq \frac{V^L - v^L}{V^H - v^H} \tag{23}
\]

(21) and (23) imply:

\[
\frac{V^L - v^L}{V^H - v^H} \leq \delta \leq \frac{v^L - \theta^L}{v^H - \theta^L}. \tag{24}
\]

If (10) holds, \( \delta < \frac{V^L - v^L}{V^H - v^H} \) which violates (24). If (11) holds, \( \frac{V^L - v^L}{V^H - v^H} > \frac{v^L - \theta^L}{v^H - \theta^L} \) which also violates (24). This eliminates case (ii). The proof is complete.

Proposition 2 provides two alternative conditions under which there is no equilibrium where high quality goods are sold and allocated to high valuation buyers. If either of these conditions are satisfied, even if high quality goods are traded, they are partly bought by low valuation buyers causing inefficiency of allocation across buyer types. The first of these two alternative conditions (condition (10)) requires that agents discount the future sufficiently. The second condition (condition (11)) is a set of restrictions on valuation parameters and under these restrictions, the conclusion of Proposition 2 holds even if the discount factor is arbitrarily close to 1. For instance, the following parameter values satisfy all assumptions made in Section 2, assumption (A1) and condition (11):

\[
V^L = 14, V^H = 30, v^L = 10, v^H = 25, \theta^L = 5, \theta^H = 24, \alpha = 0.25, \beta = 0.5
\]

Note that the conclusion of Proposition 2 allows for the possibility that no high quality goods are traded in equilibrium or that a dynamic equilibrium simply does not exist. Our next proposition outlines a stronger condition under which there exists a dynamic equilibrium where all goods are traded over time, but all high quality goods are bought by low valuation buyers. This brings out clearly the main point of this paper that while the opportunity for repeated trading and the use of waiting as a sorting mechanism may allow the volume of goods traded over time in a dynamic model to be as high as in the full information outcome, the allocation of goods across types of consumers may be quite
From assumption (6), $\alpha v^H + (1 - \alpha)v^L < \theta_H$, and under assumption $(A1)$ we have $v^H > \theta_H$. Define $\pi \in (\alpha, 1)$ by

$$\pi v^H + (1 - \pi)v^L = \theta_H. \quad (25)$$

Note $\pi$ is uniquely defined by (25).

We are now ready to state the main result:

**Proposition 3**: Assume $(A1)$. Suppose that at least one of the following holds:

(i) $\delta \leq \min\left\{\frac{V^L - v^L}{V^H - v^H}, \frac{v^L - \theta_L}{v^H - \theta_L}\right\}$ \quad (26)

(ii) for some integer $k \geq 1$ and some fraction $\pi \in [\pi, 1]$

$$\frac{v^L - \theta_L}{\pi v^H + (1 - \pi)v^L - \theta_L} \leq \delta^k \leq \frac{V^L - \theta_L}{\pi V^H + (1 - \pi)V^L - \theta_L}. \quad (27)$$

Then, there exists a dynamic equilibrium where all goods are traded, and all high quality goods are bought by low valuation buyers. This is the unique dynamic equilibrium outcome if, in addition, either (10) or (11) holds.

**Proof.** First, we show that under condition (26) the following is an equilibrium: all low quality goods are sold in period 1 and all high quality goods in period 2, all $H$-type buyers buy in period 1 while $L$-type buyers buy in periods 1 and 2. The prices are as follows:

$$p_1 = v^L, p_2 = v^H \quad (28)$$

At these prices, $L$-type buyers are indifferent between buying in period 1, buying in period 2 and not buying at all. A high quality seller would earn negative surplus if it sells in period 1 ((6) implies that $v^L < V^L < \theta_H$) and strictly positive surplus in period 2 (since $v^H > \theta_H$). A low quality seller optimally sells in period 1 if:

$$p_1 - \theta_L \geq \delta[p_2 - \theta_L]$$

i.e.,

$$(1 - \delta)\theta_L \leq p_1 - \delta p_2$$
and using (28) this holds if:

$$\delta \leq \frac{v^L - \theta_L}{v^H - \theta_L}$$

which follows from (26). Finally, H-type buyers find it optimal to buy in period 1 if:

$$V^L - p_1 \geq \delta[V^H - p_2]$$

and using (28) this is equivalent to:

$$\delta \leq \frac{V^L - v^L}{V^H - v^H}$$

which also follows from (26).

Next, we suppose that (27) holds and let $k$ and $\pi$ be as in condition (27). We construct an equilibrium some low quality goods are sold in period 1, some low and all high quality goods are sold in period $k + 1$, no trade occurs between periods 1 and $k + 1$ (if $k > 1$), all H-type buyers buy in period 1, and only L-type buyers buy in period $k + 1$. $\pi$ is the proportion of goods sold in period $k + 1$ that are of high quality; note that as $1 \geq \pi \geq \pi > \alpha$, by reducing (increasing) the amount of low quality goods traded in period $k + 1$ (period 1) appropriately one can ensure that the fraction of high quality among all goods traded in period $k + 1$ is exactly $\pi$. Set

$$p_t = \delta^k[\pi v^H + (1 - \pi)v^L] + (1 - \delta^k)\theta_L, t \leq k, p_t = \pi v^H + (1 - \pi)v^L, t \geq k + 1. \quad (29)$$

Using (25) observe that $\pi \geq \pi$ implies:

$$\pi v^H + (1 - \pi)v^L \geq \theta_H. \quad (30)$$

At prices (29), a low quality seller is indifferent between periods 1 and $k + 1$:

$$p_1 - \theta_L = \delta^k[p_{k+1} - \theta_L] = \delta^k[\pi v^H + (1 - \pi)v^L - \theta_L]. \quad (31)$$

It is easy to check that under (31), a high quality seller prefers to sell in period $k + 1$ rather than period 1. It is optimal for $H$ type buyers to buy in period 1 if

$$V^L - p_1 \geq \delta^k[\pi V^H + (1 - \pi)V^L - p_{k+1}]$$
and (using (31)) this reduces to
\[(1 - \delta^k)\theta_L = p_1 - \delta^k p_{k+1} \leq V^L - \delta^k[\pi V^H + (1 - \pi)V^L] \tag{32}\]

Further, it is optimal for \(L\) type buyers to buy in period \(k + 1\) if
\[v^L - p_1 \leq \delta^k[\pi v^H + (1 - \pi)v^L - p_{k+1}] = 0\]

and this reduces to
\[p_1 \geq v^L\]
i.e.,
\[\delta^k[\pi v^H + (1 - \pi)v^L] + (1 - \delta^k)\theta_L \geq v^L\tag{33}\]

(32) and (33) are satisfied if
\[\frac{v^L - \theta_L}{\pi v^H + (1 - \pi)v^L - \theta_L} \leq \delta^k \leq \frac{V^L - \theta_L}{\pi V^H + (1 - \pi)V^L - \theta_L} \tag{34}\]

which follows from (27). Finally, it is easy to check that no seller wishes to sell in periods strictly between periods 1 and \(k + 1\) (as prices are identical to that in period 1) and no buyer wishes to buy in those periods if they expect quality traded to be low with probability one (which is a reasonable belief as a positive fraction of low quality goods remain untraded at the end of period 1).  

Proposition 3 provides two alternative conditions under which there is an equilibrium where all goods are sold but all high quality goods are allocated to high valuation buyers. Condition (26), the first of these conditions, is satisfied as long as the discount factor is small enough. To understand the intuition behind this, suppose that \(\delta = 0\) i.e., agents are fully myopic. In that case, the market outcome in period 1 is the standard static equilibrium outcome where only low quality goods are sold and all high valuation buyers (as well as some low valuation buyers) buy. In period 2, the market is left with only high quality goods and low valuation buyers, which allows all remaining goods to be traded (as \(v^H > \theta_H\)); all high quality goods are consequently allocated to low valuation buyers. A qualitatively similar outcome results when \(\delta\) is small enough.

However, such misallocation of high quality goods to low valuation buyers does not necessarily require high degree of impatience. This is illustrated by the second alternative
condition (27) of Proposition 3; this condition may be satisfied even if the discount factor \( \delta \) is close enough to 1. The main idea is that if the equilibrium path involves sufficiently large number of intermediate periods of no trading, high valuation buyers are not willing to wait to buy high quality goods in the future. As low valuation buyers have greater incentive to wait, they end up buying the high quality goods (that can only be traded later after much of the low quality goods have been sold).

The following corollary provides a stronger but more transparent condition under which condition (27) of Proposition 3 is satisfied; this condition makes it clear that the misallocation across buyer types can arise even if discounting vanishes.

**Corollary 1** Assume (A1). Suppose that

\[
\frac{v^L - \theta_L}{v^H - \theta_L} < \frac{V^L - \theta_L}{V^H - \theta_L}. \tag{35}
\]

Then, the conclusion of Proposition 3 holds for all \( \delta \in (\delta_0, 1) \) where \( \delta_0 \in (0, 1) \) is defined by

\[
\delta_0 = \frac{v^L - \theta_L}{v^H - \theta_L}. \tag{36}
\]

**Proof.** We will show that (27) is satisfied with \( \pi = 1 \). As \( \frac{V^L - \theta_L}{V^L - \theta_L} < 1 \), \( \delta \in (\delta_0, 1) \) and (36) imply

\[
\delta > \frac{v^L - \theta_L}{v^H - \theta_L}.
\]

Therefore, if \( \delta \leq \frac{V^L - \theta_L}{V^L - \theta_L} \), (27) immediately holds (with \( k = 1, \pi = 1 \)). On the other hand, if \( \delta > \frac{V^L - \theta_L}{V^L - \theta_L} \), then let \( k > 1 \) be the smallest positive integer such that

\[
\delta^{k-1} > \frac{V^L - \theta_L}{V^H - \theta_L}. \tag{37}
\]

Then by definition,

\[
\delta^k \leq \frac{V^L - \theta_L}{V^H - \theta_L} \tag{38}
\]

and further, using (36) and (37).

\[
\delta^k = \delta^{k-1} \delta > \frac{V^L - \theta_L}{V^H - \theta_L} \delta_0 = \frac{v^L - \theta_L}{v^H - \theta_L}. \tag{39}
\]
(27) follows from (38) and (39). ■

Note that (35) is a restriction on valuation parameters that is consistent with our other assumptions. For instance, the parameter values

\[ V_L = 8, V_H = 16, v^L = 4, v^H = 11, \theta_L = 3, \theta_H = 10, \alpha = 0.20, \beta = 0.5 \]

satisfy all assumptions made in Section 2, assumption (A1) and condition (35); the conclusion of Proposition 3 holds for all \( \delta \geq \frac{13}{40} \).

5 Trading Problems

In the previous section we have seen that even though dynamic trading can expand the range of qualities traded over time and allow high quality goods to be traded eventually, there may be a misallocation of goods across buyers with high valuation buyers buying low quality goods and low valuation buyers buying high quality goods. Obviously, such an outcome requires that low valuation buyers’ willingness to pay for low quality exceed the reservation price of sellers of high quality. But what if the low valuation buyers are not willing to pay enough for high quality goods? In this section, we will show that in this situation, even though there are enough of high valuation buyers in the market to potentially buy all high quality goods (and the first best allocation is one where only high valuation buyers buy high quality goods), dynamic trading may lead to the same outcome as in a static market: only low quality goods are traded, and high quality goods are never sold. This stands in sharp contrast to the results obtained with homogenous buyers (Proposition 1) where all goods are traded eventually (in every equilibrium).

Assume that

\[(A2) \ v^H < \theta_H.\]

Assumption (A2) implies that a low valuation buyer does not buy if the price is large enough for a high quality seller to sell (even if such a buyer expects quality traded to be high with probability one).

**Proposition 4** Assume (A2). Suppose that at least one of the following holds:

\[
\frac{V^L - v^L}{V^H - \theta_H} > \frac{V^L - \theta_L}{V^H - \theta_L} > \frac{v^L - \theta_L}{\theta_H - \theta_L}
\]

(40)
\[ \delta < \min \left\{ \frac{V^L - v^L}{V_H - \theta_H}, \frac{V^L - \theta_L}{V_H - \theta_L} \right\}. \tag{41} \]

Then high quality goods are never sold in any dynamic equilibrium.

**Proof.** Suppose there is a dynamic equilibrium where both low and high quality goods are eventually traded. We will show that either,

\[ \frac{V^L - \theta_L}{V_H - \theta_L} \geq \delta \geq \frac{V^L - v^L}{V_H - \theta_H} \tag{42} \]

or,

\[ \frac{V^L - \theta_L}{V_H - \theta_L} \leq \delta^k \leq \frac{v^L - \theta_L}{\theta_H - \theta_L} \text{ for some integer } k \geq 1. \tag{43} \]

It is easy to see that if either (40) or (41) holds, then neither (42) nor (43) can hold, and this establishes the proposition\(^8\). Let \( \tau \) be the first period in which a strictly positive measure of the high quality good is traded. Then,

\[ p_\tau \geq \theta_H. \tag{44} \]

Using results R.1 and R.4 in Section 2, all low quality goods must be sold on or before date \( \tau \) and using R.3, \( \tau > 1 \). Using R.6, there is at most one period prior \( t < \tau \) in which low quality goods are traded. Using R.5.

\[ p_t \geq v^L. \tag{45} \]

Only H-type buyers buy in period \( \tau \) as (using assumption (A2) and (44)):

\[ p_\tau \geq \theta_H > v^H. \tag{46} \]

Also, low quality sellers must weakly prefer to sell in period \( t \) rather than wait for period \( \tau \),

\[ p_t - \theta_L \geq \delta^{\tau - t}(p_\tau - \theta_L) \tag{47} \]

and using (46),

\[ p_t - \theta_L \geq \delta^{\tau - t}[\theta_H - \theta_L]. \tag{48} \]

\(^8\)In fact, following the logic of the proof, it is easy to show if either (42) or (43) holds, then there is a
dynamic equilibrium where high quality goods are sold. In that sense, the conditions in Proposition 4 are
both necessary and sufficient for the conclusion.
As \( L \)-type buyers cannot buy in a period where high quality goods are sold, those that buy do so in period \( t \) and only low quality goods are sold in period \( t \). As there are more \( L \)-type buyers than low quality goods, 
\[
p_t = v^L
\]
In this equilibrium, all \( H \)-type buyers must buy. As \( H \)-type buyers definitely buy in period \( \tau \) :
\[
V^L - p_t \leq \delta^{\tau-t}[\pi_{\tau}V^H + (1 - \pi_{\tau})V^L - p_t]
\]
so that (using (49))
\[
V^L - v^L \leq \delta^{\tau-t}[\pi_{\tau}V^H + (1 - \pi_{\tau})V^L - p_t]
\]
The equilibrium can be of only two possible types: 1) low quality sellers sell in both periods \( t \) and \( \tau \), and 2) all low quality goods are sold in period \( t \). We will show that (43) holds in the former case and (42) in the latter case.
First, consider an equilibrium of type 1. In this case, (47) holds with equality so that:
\[
v^L - \theta_L = \delta^{\tau-t}(p_{\tau} - \theta_L).
\]
From (51) and (52) we have
\[
\delta^{\tau-t}[\pi_{\tau}V^H + (1 - \pi_{\tau})V^L] \geq V^L - (1 - \delta^{\tau-t})\theta_L
\]
so that
\[
\delta^{\tau-t}V^H \geq V^L - (1 - \delta^{\tau-t})\theta_L
\]
which yields:
\[
\delta^{\tau-t} \geq \frac{V^L - \theta_L}{V^H - \theta_L}
\]
Also, from (44) and (52),
\[
v^L - (1 - \delta^{\tau-t})\theta_L \geq \delta^{\tau-t}\theta_H
\]
so that
\[
\delta^{\tau-t} \leq \frac{v^L - \theta_L}{\theta_H - \theta_L}
\]
Combining (53) and (54) we have a necessary condition for an equilibrium of type 1:

\[ \frac{V^L - \theta_L}{V^H - \theta_L} \leq \delta^{\tau-t} \leq \frac{v^L - \theta_L}{\theta_H - \theta_L} \]  

(55)

i.e., (43) must hold.

Next, consider an equilibrium of type 2. Here, only high quality goods are sold in period \( \tau \). Thus, \( \pi_\tau = 1 \). Using R.6, no trade occurs after period \( \tau \). As all high valuation buyers buy and there are more of them than high quality goods, some high valuation buyers also buy in period \( t \) which implies that (50) and (51) hold with equality so that:

\[ p_\tau = V^H - \frac{V^L - v^L}{\delta^{\tau-t}}. \]

Since \( p_\tau \geq \theta_H \) we have

\[ V^H - \frac{V^L - v^L}{\delta^{\tau-t}} \geq \theta_H \]

i.e.,

\[ \delta^{\tau-t} \geq \frac{V^L - v^L}{V^H - \theta_H} \]

(56)

Also, from (49) and (47):

\[ \frac{V^L - \theta_L}{V^H - \theta_L} \geq \delta^{\tau-t}. \]

(57)

If there are intermediate periods of no trading i.e., \( \tau > t + 1 \), then as all low quality goods are sold in period \( t \), \( \pi_{t+1} = 1 \) and all high valuation buyers will strictly prefer to buy in period \( t + 1 \) if \( p_{t+1} \leq p_\tau \) (note these buyers earn strictly positive surplus). On the other hand, if \( p_{t+1} > p_\tau \), high quality sellers will strictly prefer to sell in period \( t + 1 \). Thus, it must be the case that \( \tau = t + 1 \). From (56) and (57) we then have a necessary condition for an equilibrium of type 2:

\[ \frac{V^L - \theta_L}{V^H - \theta_L} \geq \delta \geq \frac{V^L - v^L}{V^H - \theta_H} \]

(58)

i.e., (42) must hold. This completes the proof.

Proposition 4 provides two alternative conditions under which there is no equilibrium where high quality goods are traded (ever). The first of these conditions (condition (40)) is a restriction on the valuation parameters, but not on the discount factor. This condition
is consistent with our other assumptions. For instance, the parameter values
\[\begin{align*}
V^L &= 8, \\
V^H &= 16, \\
v^L &= 4, \\
v^H &= 9, \\
\theta_L &= 3, \\
\theta_H &= 10, \\
\alpha &= 0.20, \\
\beta &= 0.5
\end{align*}\]
satisfy all assumptions made in Section 2, assumption (A2) and condition (40) so that the conclusion of Proposition 4 holds for all \(\delta \in (0, 1)\). The second condition (condition (41)) in Proposition 4 requires that agents discount future payoffs sufficiently.

While Proposition 4 rules out the possibility of an equilibrium where high quality goods are traded, it does not suggest what the equilibrium outcome is, nor does it say anything about the existence of an equilibrium. The next proposition shows that under a strong condition, there exists an equilibrium where only low quality goods traded.

**Proposition 5** Assume (A2). Let \(\delta^* \in (0, 1)\) be defined by
\[
\delta^* = \max_{x \in [v^H, \theta_H]} \left[ \min \left\{ \frac{V^L - v^L}{V^H - x^*}, \frac{v^L - \theta_L}{x - \theta_L} \right\} \right] \tag{59}
\]
If
\[
\delta \leq \delta^* \tag{60}
\]
then there is a dynamic equilibrium where only low quality goods are traded and no trade occurs after period 1. This is the unique dynamic equilibrium outcome if, in addition, either (40) or (41) holds.

**Proof.** We first show that there is a dynamic equilibrium where all low quality goods are traded in period 1 and no trade occurs in later periods. Let \(x^* \in [v^H, \theta_H]\) be the solution to the maximization problem on the right hand side of (59) so that
\[
\delta^* = \min \left\{ \frac{V^L - v^L}{V^H - x^*}, \frac{v^L - \theta_L}{x^* - \theta_L} \right\}.
\]
Set \(p_1 = v^L\) and for all \(t > 1\), set \(p_t = x^*\). The expected quality is low with probability one in period 1 and in periods \(t > 1\), the expected quality is high with probability one. The sellers’ actions are as follows: all low quality sellers sell in period 1 and high quality sellers never sell. Buyers’ actions are as follows: all high valuation buyers as well as \((1 - \alpha - \beta)\) low valuation buyers buy in period 1; other low valuation buyers never buy. To see that
this is an equilibrium, note that from (60) for all \( t > 1 \),

\[
\delta \leq \min \left\{ \frac{V^L - v^L}{V^H - p_t}, \frac{v^L - \theta_L}{p_t - \theta_L} \right\}
\]

and therefore,

\[
p_1 - \theta_L = (v^L - \theta_L) \geq \delta(p_t - \theta_L) \geq \delta^t(p_t - \theta_L)
\]

so that at the chosen prices, low quality sellers find it optimal to sell in period 1; further, high quality sellers find it optimal to never sell (as they earn at most zero surplus). High valuation buyers find it optimal to buy low quality in period 1 at price \( v^L \) rather than wait to buy high quality at price \( p_t = x^* \) since

\[
V^L - p_1 = V^L - v^L \geq \delta(V^H - p_t) \geq \delta^t(V^H - p_t).
\]

Finally, low valuation buyers earn zero surplus if they buy in period 1 and non-positive surplus if they buy in any later period. This completes the proof of the first part of the proposition. Uniqueness follows from Proposition 4.

Proposition 5 provides a sufficient condition under there is an equilibrium where no high quality good is sold i.e., the market outcome is identical to that in a static model with no opportunity for dynamic sorting or revelation of information through delayed trading. This sufficient condition requires that the discount factor be relatively small - below the critical level \( \delta^* \) indicated in (60). As in the previous section, the basic intuition behind this can be best understood by considering the case where \( \delta = 0 \) i.e., all agents are perfectly myopic. In that case, the market outcome in period 1 is the standard static equilibrium outcome where only low quality goods are sold and all high valuation buyers (as well as some low valuation buyers) buy. In period 2, the market is left with only high quality goods and low valuation buyers who simply cannot trade and so no trade occurs after period 1. The outcome is qualitatively similar when \( \delta \) is relatively small. The fact that with heterogenous buyers, high quality goods may never be traded under high discounting is significant in view of the discussion at the end of Section 3 where we noted that in the homogenous consumers case, higher discounting creates more favorable conditions for trading high quality goods in the sense that delay in trading is lower.

However, note that depending on parameter values, \( \delta^* \) can be rather large. For instance,
the following parameter values:

\[ V^L = 10, V^H = 16, v^L = 4, v^H = 5, \theta_L = 3, \theta_H = 12, \alpha = 0.20, \beta = 0.5 \]

satisfy all assumptions made in Section 2 as well as assumption (A2), and for these values:

\[ \delta^* \geq \min \left\{ \frac{V^L - v^L}{V^H - v^H}, \frac{v^L - \theta_L}{v^H - \theta_L} \right\} = \frac{1}{2}. \]

6 Conclusion

The dynamic nature of durable goods markets and in particular, the repeated opportunity to trade and the possibility of waiting to trade have been viewed as important reasons why one should expect fairly brisk trading in such markets despite asymmetric information about quality between buyers and sellers. In particular, sellers of higher quality goods have greater incentive to wait and as a result, better qualities can be traded later at higher prices. The extant literature identifies delay in trading as the principal form of inefficiency. We show that when buyers are heterogenous, there is another potential source of inefficiency related to the allocation of goods across buyer types. Even if expected quality improves over time, higher valuation buyers may have greater incentive to buy earlier than buyers with lower valuation; as a result, even if all goods are traded over time, higher valuation buyers may buy lower quality leading to a loss of potential surplus. Further, once the higher valuation buyers have bought, the remaining buyers may not even have sufficient willingness to pay to buy higher quality so that trading may stop before higher quality goods are sold. Heterogeneity of buyers adds a new dimension to the nature of the "lemons problem" in dynamic markets.

Our results have been derived in a very simple framework with two quality types and two types of buyers. It is intuitively clear that qualitatively similar results should hold for more general discrete distributions of quality and buyer types. The basic tension between the incentive to wait of various types buyers and that for sellers of different qualities is a more general phenomenon and one that has not been sufficiently exploited in the existing theoretical literature on various mechanisms of dynamic trading under asymmetric information. In particular, our results about the possibility of inefficient intertemporal allocation of quality across buyer types can hold even with a continuum of quality and
buyer types; they should also hold for markets with decentralized trading and frictions (as well as non-market mechanisms such as bargaining and auctions).

References


