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RESEARCH ARTICLE

Irreversibility and the economics of forest conservation

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Abstract

Regenerating forest on land used for non-forest economic activities can be difficult; this introduces some irreversibility in the process of deforestation. We analyze the effect of such irreversibility (reforestation cost) on the efficiency of forest conservation in a general model of optimal forest management where trees are classified in age classes and land has alternative economic use. Irreversibility may lead to a continuum of optimal steady states that differ in the area under forest cover; increase in irreversibility can only add steady states with smaller forest cover. High irreversibility discourages expansion of forests but at the same time, makes it optimal to conserve a minimal forested area in the long run; in particular, it is optimal to maintain a forested area above a critical size if the initial forest cover lies above it, while forests that are initially smaller than the critical level are optimally managed at constant size. We characterize the exact condition under which it is optimal to avoid total deforestation; the extent of irreversibility does not matter for this. Weak irreversibility may be associated with cyclical fluctuations in optimal forest cover; we characterize upper and lower bounds on the forest cover along an optimal path.

Keywords Irreversibility · Deforestation · Conservation · Optimal forest management · Renewable resources

This paper is dedicated to Professor Tapan Mitra on the occasion of his 70th birthday; the beauty and elegance of his fundamental work on some of the most challenging problems in the economic theory of dynamic resource allocation inspires and paves the path for generations of researchers. Adriana Piazza gratefully acknowledges the financial support of FONDECYT Project 1180409 and Basal Project CMM, U. de Chile and the hospitality of Southern Methodist University during several research visits. We are grateful to two anonymous referees for their helpful comments and suggestions.

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1 Introduction

Deforestation is an important environmental concern. The Global Forest Resources Assessment 2015 of the Food and Agricultural Organization of the United Nations (FAO 2015) indicates that the average rate of global forest loss for the period 2010–2015 is 3.3 million hectares each year (about 0.08% of the total forest area). Though the rate of deforestation is alarmingly high, it is about half of the rate observed in the 1990s. Encouragingly, an expansion of forest area is observed in some regions where such area had been shrinking until recently, indicating that forest depletion may have been excessive in the past.[1] Social scientists tend to focus on tropical deforestation,[2] where weakness of property rights (leading to encroachment and illegal logging) and myopic management practices are some of the key human factors.[3] However, even if property rights are well defined and well enforced and the forest is managed by a forward looking private owner or a public agency, it may make sense to divert forest land to other economic uses such as agriculture, mining and construction; deforestation in the sense of a sustained decline in forest cover can be the consequence of an optimal strategy where trees are cut down without replanting and forest land is diverted to alternative uses over time. It is therefore important to understand the factors that determine the dynamic economic incentives for long-run conservation of forest cover in optimally managed forests.

An important factor here, and one that has not been adequately emphasized in the extant literature on optimal forest management, is that deforestation can often be a somewhat irreversible process; there are additional costs associated with regenerating forest in land that is currently utilized for other purposes because of soil degradation and erosion, damage to the local ecology and biodiversity, changes in local climate and so on.[4] We refer to this additional cost as reforestation cost.[5,6] Reforestation cost is likely to depend on the nature of alternative economic activities for which land has been used (as well as the kind of forest under consideration). It affects the dynamic incentives for conversion of forest land to alternative use along with other pertinent factors such as the age structure of trees in the forest, the flow benefits from harvesting timber, the size of the area under forest cover and the returns from alternative use. This paper attempts to analyze rigorously how irreversibility (i.e., reforestation cost) affects forest conservation and deforestation when both the area of land under forest cover and the age structure of the forest are optimally managed over time.

[1] See the review and synthesis in Sloan and Sayer (2015) based on data presented in FAO (2015).

[2] There is a large literature on tropical deforestation (see, for instance, Arild and Kaimowitz 1999); this literature ignores the age structure of forests.

[3] For an excellent exposition of the literature, see Amacher et al. (2009) .

[4] See, for instance, Dupouey et al. (2002) and Rnyan et al. (2012)

[5] The *average* cost of establishing forest vegetation was estimated as $270 per acre in the Fiscal Year 2017 Budget Justification of the US Department of Agriculture (p. 377). Depending on the nature of current use of land, the actual number can be significantly higher.

[6] This cost is not incurred when trees are harvested and replanted on land currently under forest cover

In the existing literature, conditions for optimal extinction and conservation have been characterized for renewable resources whose biological growth is determined mainly (if not exclusively) by the size of the remaining population.[7] Forests are however somewhat different from many other renewable resources in that regeneration is largely dependent on the availability of land and on decisions regarding land use. Models of forest management also need to take into account the relatively long rotation, the multi-age structure and the age-dependent timber content of trees. In managed forests, trees can always be planted and zero resource stock is not an absorbing state. As a result, the concept of "extinction/conservation" of a forest is somewhat different from that concerning other biological species. Total deforestation (the analog of extinction) can be said to occur when all available land is diverted to alternative (non-forest) use in the long run.[8] Conservation then refers to sustaining strictly positive forest cover in the long run.

We consider a general discrete time model of optimal forest management which is a variation of the well-known model due to Mitra and Wan (1985, 1986) where it is assumed that the timber content per unit of forest area is related only to the age of the trees, so that the forest can be represented as a collection of age classes.[9] The focus of the Mitra-Wan papers (and indeed of much of the subsequent theoretical literature on optimal forest use, see, for example, Salo and Tahvonen 2003 and Tahvonen 2004) is on the dynamics of forest rotation, and therefore, it is assumed that the total area under forests is fixed over time. Salo and Tahvonen (2004) introduce the possibility of alternative use of land in the Mitra-Wan framework; they focus on the existence and uniqueness of (optimal) steady state. They show the existence of optimal periodic cycles when the steady state is a pure forestry state (with no land in alternative use), and the impossibility of such cycles if the steady state is "mixed"; they also analyze the stability of the steady state in the latter case.[10] Piazza and Roy (2015) analyze the problem of optimal depletion of forest area in this augmented Mitra-Wan framework (with alternative use of land); the paper characterizes fully the conditions for optimality of total deforestation as well as lower bounds on the forest cover along the optimal path.

Salo and Tahvonen (2004) first introduced a cost of conversion of land from forestry to alternative use (and vice versa) in a model of forest rotation with alternative use of land. The paper demonstrated numerically the possibility of a continuum of steady states and the (path) dependence of long-run equilibrium land allocation on the initial state. The paper also contains examples where the long-run equilibria are cyclical solutions around the normal forest steady states. Our paper follows up on some of the ideas and the modeling approach explored in that paper and develops a general characterization. In particular, we introduce irreversibility in forest depletion into the augmented Mitra-Wan model of optimal forest management with alternative use of land by allowing for a reforestation cost that is incurred in the period in which land

7 See, for instance, Clark (2010).

8 In contrast to "total deforestation," the term "deforestation" will be used to refer to any decline in forest cover.

9 This assumption may not be applicable to wild forests.

10 However, they do not study the existence or stability of a steady state where land is allocated exclusively to alternative use, and therefore, their analysis does not shed light on total deforestation.

is converted from alternative use to forestry. The total reforestation cost depends on the amount of land that is reforested in that period. The marginal cost of reforestation captures the extent of irreversibility in deforestation. To focus specifically on the consequence of damage to suitability of land for forestry that is caused by non-forest use, we assume that there is no cost associated with conversion of land from forestry to alternative use; further, we ignore the cost of replanting trees in areas currently under forest cover.

Irreversibility has two effects on the dynamic incentives for conservation of forests. There is a direct effect: reforestation being difficult reduces the incentive to expand forest cover. However, there is an opposing indirect effect: anticipating the difficulty of future reforestation may reduce the current incentive to deforest. The results in this paper bring out the net consequence of these two effects on various aspects of forest conservation.

First, an increase in the extent of irreversibility adds to the set of optimal steady states but the added steady states are characterized by lower forest cover. In the absence of irreversibility (no reforestation cost), the convex structure of the model (and in particular, strict concavity of the benefit from harvesting timber) ensures that there is a unique optimal steady state. We show that introducing irreversibility can lead to a continuum of optimal steady states that differ in the area under forest cover (though all of them have identical "balanced" distribution of the forested area across age classes). One implication of this is that under irreversibility, the long-run forest cover depends on the initial state of the forest. The set of areas under forest cover sustained in optimal steady states is a closed interval. An increase in irreversibility (i.e., marginal reforestation cost) can only reduce the lower bound of this interval; sufficient increase in irreversibility may reduce the lower bound to zero. Thus, high irreversibility makes it more likely that a small forest area with a balanced age structure is sustained as a steady state over time.

Second, if irreversibility is high enough (i.e., the *intrinsic* marginal reforestation cost is above a certain level) then starting from a strictly positive initial forest cover, the forest cover along an optimal path is bounded away from zero; we call this *strong conservation*. In particular, we show that under high irreversibility there is a certain threshold size of forest area such that starting from any initial forest area above the threshold, the area under forest cover along an optimal path never declines to a level below the threshold; further, if the initial forest cover is below this threshold, then the optimal forest cover is unchanged over time (though the age structure of trees may change over time). Thus, high irreversibility not only ensures that strong conservation of forests is optimal, but also that small forest areas are likely to be maintained with no deforestation over time. These effects reflect the indirect effect of irreversibility mentioned above. However, high irreversibility also implies that reforestation is never optimal, i.e., the area under forest cover never expands over time; this reflects the direct effect of irreversibility.

Third, we consider *weak conservation* defined as the avoidance of total deforestation from an initial state with strictly positive forest cover. We derive a condition that is both necessary and sufficient for weak conservation. This condition is independent of the level of irreversibility; in particular, it is identical to the condition for avoidance of total deforestation in the absence of irreversibility. The implication is that the extent of irreversibility, i.e., marginal reforestation cost, does not matter for whether or not it is optimal to reduce forest cover to zero in the long run. Now, it is somewhat intuitive that a path where the forest cover converges to zero in the long run is unlikely to be characterized by any reforestation, and if such a path is optimal in the absence of reforestation cost then it is likely to be optimal even when reforestation is costly. What is surprising is that the converse is also true; if a path where the forest cover disappears over time is optimal with high reforestation cost, it continues to be optimal with low reforestation cost. Under the conditions for weak conservation, optimal paths may be characterized by fluctuations, possibly cyclical, in the area under forest covers over time. We characterize uniform upper and lower bounds on the forest cover along an optimal path; these bounds are independent of the extent of irreversibility.

However, the extent of irreversibility does matter for whether or not weak conservation is optimal from an initial state with zero forest cover (all land is in alternative use); more generally, irreversibility matters for whether or not it is optimal to regenerate a forest from such an initial state. Even if forest regeneration is optimal when irreversibility is low, high irreversibility can prevent regeneration making the state of zero forest cover an optimal steady state. Here, the direct effect of reforestation cost hinders expansion of forests.

Our analysis of forest cover along optimal paths and the effect of irreversibility is considerably complicated by rotation in the age structure of the forest and the possible cyclical variation in the area under forest. In the special case of a single-age forest (where all trees are harvested at the end of each period and the state variable is a scalar), there are no cycles and optimal forest cover is monotonic over time. In this case, we obtain a much tighter characterization of conservation and optimal paths. While some of the general effects of increase in irreversibility such as expansion of the set of optimal steady states to include states with smaller forest cover continue to hold in this special case, irreversibility appears to matter far less for conservation than in a multi-age forest structure.

The effect of irreversibility on accumulation of capital has been widely addressed in the optimal growth literature since the seminal papers of Cass (1965) and Arrow and Kurz (1970). In this literature, irreversibility is related to the difficulty in conversion of capital goods to consumption and generally takes the form of a lower bound on the extent to which capital can be immediately depleted (for instance, through depreciation). If irreversibility is less than full, i.e., some depreciation or downward adjustment of capital stock is allowed for, then the economy always adjusts toward an optimal steady state that is independent of irreversibility; the latter does not affect the set of optimal steady states or the long-run destiny of the economy (only affects the transition path). This is very different from the results in our framework. Also, unlike the optimal growth models, we introduce irreversibility not as a constraint on state transition but rather as an immediate cost that discourages forest expansion, i.e., restrains accumulation rather than depletion of "natural" capital.[11]

[11] Our analysis of the single-age forest case in Sect. 7 provides some indication of the extent to which the qualitative effects of increase in irreversibility identified in this paper may extend to "Ramsey" models of optimal capital accumulation with a scalar state variable where irreversibility enters as a cost of conversion of sector-specific capital.

We deliberately choose to abstract from issues related to uncertainty (for instance, about future benefits) in order to keep the analysis tractable. This allows us to obtain a clean characterization of the primary effects of irreversibility on forest conservation. This contrasts with a significant literature on irreversibility that began with the classic papers by Arrow and Fisher (1974) and Henry (1974) examining dynamic decision making where a current action has irreversible effects (for instance, development of a tract of land that can lead to irreversible loss of environmental benefits) and there is uncertainty about future preferences (future benefits from preserving the piece of land). The focus of this literature is comparison across information structures; a key result is that under certain conditions on the preferences, knowing that better information about future preferences will be available later makes it optimal to make current decisions that reduce the irreversible effects and in particular, creates an option value of preserving environmental assets (see also, Fisher and Hanemann 1986 and the excellent survey by Mäler and Fisher 2005). Though our framework is deterministic, we find that the difficulty of future reforestation creates incentive to conserve current forests; this effect is somewhat similar to the precautionary motive for preserving environmental assets in the classical literature on irreversibility and uncertainty. Albers et al. (1996) also analyze issues of uncertainty and irreversibility in a reduced form model of tropical deforestation (see also, Albers 1996); there is no issue of optimal dynamic management of forest structure in these papers. In contrast to this literature, our model is fully deterministic but involves a more complete dynamic model of irreversible deforestation with endogenous forest structure.[12]

Section 2 outlines the model. Section 3 contains our characterization of optimal steady states and the effect of irreversibility. Section 4 characterizes the condition for strong conservation (where the extent of irreversibility plays an important role) and characterizes a strictly positive uniform lower bound on the area under forest cover along an optimal path. Section 5 develops a tight condition for weak conservation and a bound that the area under forest cover must exceed from time to time. Section 6 outlines some upper bounds on the forest cover that potentially depend on the extent of irreversibility. Section 7 analyzes the special case of a single-age forest. Section 8 illustrates our theoretical results for the general model in the context of a numerical example. Appendix A outlines modified versions of some of our results when the assumption on the benefit function is weakened from strict concavity to weak concavity. Appendix B contains proofs of all results and a lemma that is used in the proofs.

2 The model

Our model is an extension of the well-known model of evolution and management of a multi-age forest due to Mitra and Wan (1985, 1986); we augment the model to allow

[12] There is also a significant literature on the effect of irreversibility in capital formation (less than full depreciation) on economic growth and business cycles; see, among others, Sargent (1980); Olson (1989); Majumdar and Nermuth (1982); Dow and Olson (1992) and the survey by Olson (1996). The underlying framework in this literature is one (or two) sector model of aggregative growth which is significantly different from the optimal forest rotation model.

for diversion of land to alternative use (as in Piazza and Roy 2015) and, in addition, introduce irreversibility in deforestation through a positive cost of reforestation.

Time is discrete and is indexed by $t = 0, 1, 2, \ldots$. The total area of land is constant over time and set equal to 1. Land is allocated between forest and an alternative use. The forest consists of trees whose age can vary from 1 to n where $n > 1$ represents the age at which a tree dies or loses its economic value. The timber content per unit of forest area is related only to the age of the trees. Thus, one can group the trees into n age classes and represent the state of the forest by the area occupied by each age class.[13] More precisely, let $x_{a,t}$ denote the area occupied by the output of trees of age a at the very beginning of period t *before any harvesting or planting of trees take place*. In each period t, the state of the forest can be represented by the vector $\boldsymbol{x}_t = (x_{1,t}, \ldots, x_{n,t})$,[14] where $\boldsymbol{x}_t$ belongs to the set $\mathcal{D}$ defined by

$$\mathcal{D} = \left\{ \boldsymbol{x} \in \mathbb{R}_+^n : \sum_{a=1}^{n} x_a \leq 1 \right\},$$

and the total forest cover is

$$z_t = \sum_{a=1}^{n} x_a \tag{1}$$

where $z_t \in [0, 1]$ for all t. We assume that all land not occupied by the standing forest has been dedicated to alternative use. The area under alternative use at the beginning of period t (or more precisely, during the full length of period $(t-1)$) is then given by $y_t = 1 - z_t$.

Given the current state $\boldsymbol{x}_t$, two sets of decisions are made by the forest owner (or manager) at the beginning of period t. First, she must decide on $c_{a,t} \in [0, x_{a,t}]$, the current harvest of trees of age class a or more precisely, the part of the land occupied by (the output of) trees of age class a that is harvested in that period, $a = 1, \ldots, n$. The size of current harvest at the beginning of period t is then $\boldsymbol{c}_t = (c_{1,t}, \ldots, c_{n,t})$. Without loss of generality, we assume that $c_{n,t} = x_{n,t}$. After this harvest,

$$\tilde{x}_{a,t} = x_{a,t} - c_{a,t} \tag{2}$$

is the area occupied by trees of age class a that remain standing, $a = 1, \ldots, n$. Second, given the harvesting decision $\boldsymbol{c}_t$, the total land available for replanting is $\sum_{a=1}^{n} c_{a,t} + y_t$; of this area, an area $\tilde{x}_{0,t} \in \left[0, \sum_{a=1}^{n} c_{a,t} + y_t\right]$ is planted with new seedlings and the rest (denoted by y_{t+1}) is diverted to alternative use during the full length of the period t. With the understanding that age 0 represents new seedlings and that $\tilde{x}_{n,t} = 0$, the

[13] Mitra and Wan (1985, 1986) take n to be the age at which the biomass per unit of land is maximized. As pointed out in Khan and Piazza (2012), concavity of the benefit function then favors a homogeneously configured forest and it may be optimal to postpone harvesting beyond age n in order to reshape the forest into a more homogeneous state. Following the approach used in Khan and Piazza (2012), we circumvent this issue by assuming n to be the age after which a tree dies.

[14] The expressions in bold print represent vectors.

vector $(\tilde{x}_{0,t}, \tilde{x}_{1,t}, ...\tilde{x}_{n-1,t})$ then gives us the land occupied by the *input* of trees of all age classes at the beginning of period *t after harvesting and replanting decisions are made*. This input generates the output of trees of various age classes at the beginning of period $t+1$ through the natural production process of aging. In particular, the area occupied by (the output of) trees of age $(a+1)$ at the very beginning of period $t+1$ is given by

$$x_{a+1,t+1} = \tilde{x}_{a,t},\ a = 0, 1, ..., n-1. \tag{3}$$

At the very beginning of period $t+1$ (prior to any harvesting and replanting), the agent faces a standing forest

$$\boldsymbol{x}_{t+1} = (x_{1,t+1}, \ldots, x_{n,t+1}) = (\tilde{x}_{0,t}, x_{1,t} - c_{1,t}, \ldots, x_{n-1,t} - c_{n-1,t})$$

and an amount $y_{t+1} = 1 - \sum_{a=1}^{n} x_{a,t+1}$ of land that has been dedicated to alternative use (over the full length of the period t).

A sequence $\{\boldsymbol{x}_t\}_t$ is a *program* iff

$$\boldsymbol{x}_t \in \mathcal{D}\ \forall t \ \text{ and } \ x_{a+1,t+1} \leq x_{a,t} \ \text{ for } a = 1, \ldots, n-1\ \forall t$$

Associated with any program, we have a sequence of land dedicated to the alternative use, $\{y_t\}$, a sequence of total area under forest cover $\{z_t\}$ and a sequence of harvests, $\{\boldsymbol{c}_t\}$, that are calculated using (1), (2) and (3); we refer to $\{y_t, z_t, \boldsymbol{c}_t\}$ as a *path* associated with program $\{\boldsymbol{x}_t\}$.

As in the original model analyzed by Mitra and Wan (1985, 1986), the timber content of one unit of area covered by trees of age a is given by a production function $f(a) \geq 0$. For notational convenience, we define the biomass coefficients as $f_a = f(a)$ for $a = 1, \ldots, n$.[15] The current *timber* consumption in period t generated by the current harvest of trees $\boldsymbol{c}_t = (c_{1,t}, \ldots, c_{n,t})$ is denoted by C_t and is given by

$$C_t = \sum_{a=1}^{n} f_a c_{a,t} = \sum_{a=1}^{n-1} f_a (x_{a+1,t+1} - x_{a,t}) + f_n x_{n,t}. \tag{4}$$

After tree cutting at the beginning of period t, the forest manager receives net benefit from timber consumption, $u(C_t)$; the net benefit may represent current profit for a private forest owner, the current social surplus for a planner or some other objective function used by the forest manager. In addition, at the beginning of period t, the forest manager receives a return $w(y_t)$ from the dedication of land y_t to alternative use over the length of the previous period $(t-1)$.

To capture irreversibility in the removal of forest cover and in particular, the fact that non-forest economic activities including agriculture, construction and manufacturing may cause ecological degradation of land and make it somewhat unsuitable

[15] As in Khan and Piazza (2012), we dispense with most of the restrictions on the biomass coefficients that are found in the literature, for instance, in Mitra and Wan (1985, 1986); Salo and Tahvonen (2003, 2004) and Tahvonen (2004).

for reforestation, we introduce a cost of reforestation $\nu(\lambda)$ associated with converting an area of land $\lambda \geq 0$ from alternative use to forestry. This cost of reforestation is incurred only in the period in which land is diverted from alternative use to forestry. The "intrinsic" marginal cost of reforestation (when there is no reforestation activity in the current period) will be denoted by $\delta \geq 0$. In reality, there may also be specific economic costs associated with conversion of land from forestry to alternative use but we ignore these in order to focus exclusively on the difficulty of reforestation. In particular, we assume that diversion of land from forestry to alternative use is cost-less.

The following assumptions are imposed:

A_1 : $u : \mathbb{R}_+ \to \mathbb{R}$ and $w : [0, 1] \to \mathbb{R}_+$ are non-decreasing, continuous on $\mathbb{R}_+$ and differentiable on $\mathbb{R}_{++}$. w is concave.

A_2 : u is strictly concave.

A_3 : $\nu : \mathbb{R} \to \mathbb{R}_+$ is continuous, convex and non-decreasing; $\nu(\lambda) = 0$ for $\lambda \leq 0$. Further, $\nu(\lambda)$ is differentiable on $\mathbb{R}_{++}$ and $\nu'_+(0) = \lim_{\lambda \downarrow 0} \nu'(\lambda) = \delta \geq 0$.[16]

A_4 : there is a *unique* $\sigma \in \{1, \ldots, n\}$ such that

$$\frac{b^\sigma f_\sigma}{1 - b^\sigma} \geq \frac{b^a f_a}{1 - b^a} \quad \text{for all } a = 1, \ldots, n. \tag{5}$$

Note that $\frac{b^a f_a}{1-b^a}$ is the discounted sum of intertemporal timber consumption (from a forest land of unit size) under a policy where newly planted trees are left to grow until they reach age a and fully harvested after every a periods; harvesting after every σ periods maximizes the intertemporal timber consumption within this class of policies. It is worth observing that σ depends only on the primitives of the model: the discount factor b and the biomass coefficients f_a. In particular, it does not depend on the actual area occupied by the forest.

Given an initial state $\mathbf{x} \in \mathcal{D}$, the forest manager solves the following dynamic optimization problem:

$$\begin{cases} \text{maximize} & \sum_{t=0}^{\infty} b^t [u(C_t) + w(1 - z_t) - \nu(z_{t+1} - z_t)] \\ \text{subject to} & (1) \text{ to } (4) \\ & \{\boldsymbol{x}_t\} \text{ is a program and } \boldsymbol{x}_0 = \mathbf{x} \end{cases} \tag{6}$$

where $0 < b < 1$ is the discount factor, $\boldsymbol{c}_t$ and y_t are the control variables and $\boldsymbol{x}_t$ is the state variable.

We say that the program $\{\boldsymbol{x}_t\}$ is an *optimal program* from the initial state $\mathbf{x}$ if it solves the optimization problem (6). In addition, we say that $\mathbf{x}$ is an *optimal steady state (oss)* if the optimal program from $\mathbf{x}$ is such that $\boldsymbol{x}_t = \mathbf{x}$ for all t.

We shall refer to the path associated with any optimal program as an *optimal path*.

Section A of the Appendix indicates how some of the results are modified when A_2 is replaced by a weaker concavity requirement. Assumption A_4 ensures that in any optimal steady state, only trees of age σ are harvested along the stationary optimal path.

16 When $\delta > 0$, the function ν is not differentiable at 0. As ν is convex (and finite valued), the one-sided derivatives are well defined at zero.

3 Optimal steady states

In this section, we fully characterize the set of optimal steady states (hereafter, oss) and study their variation with respect to irreversibility, i.e., reforestation costs.

Recall that $\delta = \nu'_+(0)$ represents the "intrinsic" marginal cost of reforestation of land under alternative use. As it turns out, changes in the cost of reforestation affect the optimal steady states only through the parameter δ as there is no actual reforestation activity along a steady-state path. In particular, the set of optimal steady states is unaffected by a shift in the reforestation cost function $\nu(z)$ that leaves δ unchanged.

In what follows, we focus on variation in the degree of irreversibility, exclusively in terms of changes in the intrinsic marginal reforestation cost $\delta = \nu'_+(0)$. In particular, when we refer to higher or lower δ we mean a shift in the ν function that changes its right-hand slope at zero.

We begin with an important observation:

Lemma 1 *Given $\widehat{\delta} \geq 0$, if $\boldsymbol{x}$ is an oss when $\delta = \widehat{\delta}$ then it is also an oss for every $\delta > \widehat{\delta}$.*

Proof First note that while a stationary path associated with a steady state can, in principle, involve conversion of land (while keeping the total forest area constant), such a stationary path can be optimal only if no reforestation cost is incurred along the path[17] and in that case, there exists an equivalent stationary path with exactly identical forest structure and forest cover where there is no conversion of land. Therefore, without loss of generality, the stationary optimal path associated with oss $\boldsymbol{x}$ at $\delta = \widehat{\delta} \geq 0$ can be taken to be one with no conversion of land. An increase in the cost of reforestation does not affect the total discounted utility obtained along such a path. Further, it is not possible that with increase in reforestation cost some other program leads to higher total discounted utility. Thus, $\boldsymbol{x}$ continues to be an oss for every $\delta > \widehat{\delta}$. □

This lemma has two important implications.

Corollary 1 *The set of oss can only expand as δ increases. In particular, any oss in the absence of reforestation cost remains an oss when such costs are introduced.*

Let $\boldsymbol{x}(z)$ denote a specific state where $z \in [0, 1]$ is the total area of land under forest cover and this area of land is distributed evenly among trees of age classes 1 through σ, where σ is defined by (5) in A_4, i.e.,

$$\boldsymbol{x}(z) = \Big(\underbrace{\frac{z}{\sigma}, \ldots, \frac{z}{\sigma}}_{\sigma}, \underbrace{0, \ldots, 0}_{n-\sigma}\Big)$$

The next lemma shows that every oss must have this structure.

[17] If reforestation cost is incurred along the stationary path, then as the forest size must remain constant over time, it must be the case that conversion of any area of forested land to alternative use, must be matched by costly conversion of an equivalent area to forestry. As land is assumed homogeneous, no additional benefit can be obtained by this area swap and the net effect on the forest manager's utility is negative.

Lemma 2 *If $\boldsymbol{x}$ is an oss, it must be of the form $\boldsymbol{x}(z)$ for some value of $z \in [0, 1]$.*

To determine the actual size of the forest cover in an oss, we introduce the following function that captures the marginal benefit from reforestation along a steady-state path:

$$\Delta : [0, 1] \to \mathbb{R}$$
$$\Delta(z) = \frac{b^{\sigma} f_{\sigma}}{1 - b^{\sigma}} u'\left(\frac{f_{\sigma}}{\sigma} z\right) - \frac{b}{1 - b} w'(1 - z).$$

Specifically, if the current state is $\boldsymbol{x}(z)$, $\Delta(z)$ represents the marginal benefit of dedicating more land to forestry at $t = 0$ if there is no further reallocation of land in the future and only σ-age class trees are harvested over time. It is easy to check that under our assumptions, Δ is continuous and strictly decreasing on $[0, 1]$. The function $\Delta(\cdot)$ plays a very important role in our results.

Let the forest size $z_0 \in [0, 1]$ be defined by:

$$z_0 = \begin{cases} \Delta^{-1}(0), \text{ if } \Delta(1) \le 0 \le \Delta(0) \\ 1, \text{ if } \Delta(1) \ge 0 \\ 0, \text{ if } \Delta(0) \le 0. \end{cases}$$

Lemma 3 *$\boldsymbol{x}(z_0)$ is the unique oss when $\delta = 0$ (and in particular, when there is no reforestation cost); further, it is an oss for every $\delta \ge 0$.*

It immediately follows that

Corollary 2 *If $\boldsymbol{x}$ is the unique oss for any $\delta \ge 0$, then $\boldsymbol{x} = \boldsymbol{x}(z_0)$.*

The above results indicate that irreversibility can make a qualitative difference to the nature of oss only by introducing multiplicity of such states.

Given δ, let the forest size $z_\delta \in [0, 1]$ be defined by:

$$z_\delta = \begin{cases} \Delta^{-1}(\delta), \text{ if } \Delta(1) \le \delta \le \Delta(0) \\ 1, \text{ if } \Delta(1) \ge \delta \\ 0, \text{ if } \Delta(0) \le \delta. \end{cases}$$

It is easy to check that

$$z_\delta \le z_0.$$

Further, for $\delta > 0$

$$z_0 = z_\delta = 0 \text{ if, and only if, } \Delta(0) \le 0$$
$$z_0 = z_\delta = 1 \text{ if, and only if, } \Delta(1) \ge \delta$$

and

$$z_\delta < z_0 \text{ if, and only if, } \Delta(0) > 0 \text{ and } \Delta(1) < \delta.$$

Finally, let $I(\delta)$ be the set of forest sizes sustained along oss paths:

$$I(\delta) = \left\{z \in [0, 1] : \boldsymbol{x}(z) \text{ is an oss given } \delta = \nu'_+(0)\right\}.$$

Proposition 1 *The set of oss coincides with* $\{\boldsymbol{x}(z) : z \in [z_\delta, z_0]\}$, *i.e., the set of states with forest cover* $z \in [z_\delta, z_0]$ *where the forest area* z *is evenly divided among trees of age classes* 1 *through* σ. *In particular,*

1. *If* $\Delta(0) \leq 0$, *then* $I(\delta) = \{0\}$ *for all* $\delta \geq 0$, *i.e., regardless of the extent of irreversibility (cost of reforestation) the totally deforested state is the unique oss.*
2. *If* $\Delta(1) \geq \delta$, *then* $I(\delta) = \{1\}$, *i.e., there is a unique oss and it is characterized by full forest cover.*
3. *If* $\delta > 0$, $\Delta(0) > 0$ *and* $\Delta(1) < \delta$, *then* $z_\delta < z_0$ *and there is a continuum of oss; the set of forest covers that can be sustained in an oss is a closed interval* $[z_\delta, z_0]$ *with non-empty interior. In addition,* $I(\delta) = \{z \in [0, 1] : 0 \leq \Delta(z) \leq \delta\}$.

The most interesting result in Proposition 1 is the non-uniqueness of oss even though the benefit function is strictly concave and A_4 holds. To the best of our knowledge, this is the first analytic proof of the existence of a continuum of oss in models of optimal forest management. This possibility was first shown by Salo and Tahvonen (2004) in a numerical example. The non-uniqueness of oss clearly establishes the dependence of long-run behavior on the initial state of the forest.

The oss is unique, regardless of the marginal reforestation cost, if $\Delta(0) \leq 0$; in that case, marginal benefit from reforestation along a steady-state path is always negative so that total deforestation is optimal and a steady-state path must have zero forest cover. If $\Delta(0) > 0$, the oss can still be unique if $\Delta(1) \geq \delta$, i.e., the marginal benefit from reforestation along a steady-state path with less than full forest cover exceeds the intrinsic marginal reforestation cost and it is optimal to expand the forested area; the oss must then be the one with full forest cover.

However, if $\delta > 0$, $\Delta(0) > 0$ and $\delta > \Delta(1)$, then we have a continuum of oss with $\boldsymbol{x}(z_0)$ being the oss with highest forest cover.

To understand this, recall that (as noted in Lemma 3) $\boldsymbol{x}(z_0)$ is the unique oss when there is no reforestation cost and it is always an oss for $\delta > 0$. In particular, as $\Delta(0) > 0$ the marginal benefit from reforestation along a steady-state path with no forest cover is strictly positive; so when there is no reforestation cost, there is a strict incentive to move away from the zero forest cover steady state which implies that $z_0 > 0$. Now, suppose that $z_0 < 1$. Then, by definition, $\Delta(z_0) = 0$ so that $\delta > 0$ implies that the intrinsic marginal reforestation cost is higher than the marginal benefit from reforestation along the steady state $\boldsymbol{x}(z_0)$. Therefore, in a candidate steady-state path with forest cover z that is slightly lower than z_0 there is no incentive to add to the forest area through reforestation; further, as $z < z_0$ and $\boldsymbol{x}(z_0)$ is an oss, the monotonicity of $\Delta(\cdot)$ ensures that there is no incentive to further reduce forest cover from the candidate steady-state path $\boldsymbol{x}(z)$. Thus, $\boldsymbol{x}(z)$ is an oss for z slightly lower than z_0.

The smallest forest cover that can be sustainable as an oss is z_δ where the marginal benefit from reforestation equals the intrinsic marginal cost of reforestation; in candidate steady-state paths with forest cover smaller than z_δ (assuming $z_\delta > 0$) there would be a strict incentive for expansion of forest area. Note that there cannot be an oss $\boldsymbol{x}(z)$ with $z > z_0$ as it is not an oss in the case of no reforestation cost which means there is a strictly positive gain from lowering the forest area. (The presence of reforestation cost is not relevant for the payoff of reducing forest cover.) This yields part 3. of Proposition 1. However, this explanation runs into a potential problem when we have a corner solution for z_0 with $z_0 = 1$; in that case if $\delta \leq \Delta(1)$ there is incentive to increase the forest area from a steady-state path with less than full forest cover and so a candidate steady-state path with forest cover z slightly below $z_0 = 1$ would not be sustainable as an oss. The condition $\Delta(1) < \delta$ rules out this possibility and guarantees that there are oss with forest area below z_0 even when $z_0 = 1$.

The following corollary provides further information about the sensitivity of the upper and lower bounds on optimal steady states with respect to reforestation cost:

Corollary 3 *Assume that* $\Delta(0) > 0$.

1. $\boldsymbol{x}(z_0)$, *the oss with the highest forest cover, is unaffected by reforestation cost; it is characterized by full forest cover* ($z_0 = 1$) *if, and only if,* $\Delta(1) \geq 0$.
2. *If* δ, *the intrinsic marginal reforestation cost, increases from 0 to* $\Delta(0)$, *then* $I(\delta) = [z_\delta, z_0]$ *expands from* $I(0) = \{z_0\}$ *to* $I(\Delta(0)) = [0, z_0]$ *as* z_δ *declines from* z_0 *to* 0, *i.e., set of oss strictly expands with increasing* δ *but the added steady states have strictly lower forest cover.*
3. *If* $\delta \geq \Delta(0)$, *then* $I(\delta) = [0, z_0]$, *i.e., the fully deforested state is an oss and further increase in reforestation cost has no effect on the set of optimal steady states.*

The proof of the Corollary follows directly from the fact that $\Delta(z)$ is strictly decreasing.

Proposition 1 and Corollary 3 are illustrated in Figs. 1, 2 and 3.

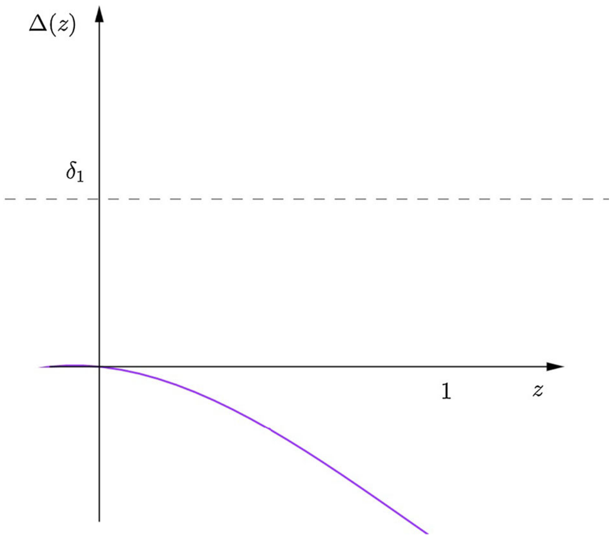

Fig. 1 $I = \{0\}$ for all values of δ

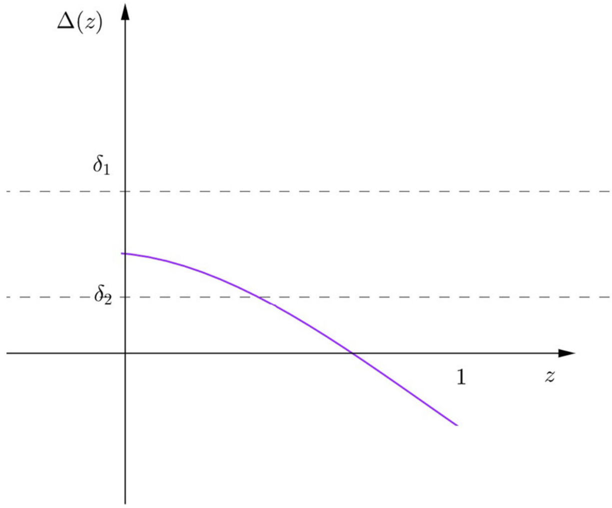

Fig. 2 If $\delta = \delta_1$ then $I = [0, z_0]$, if $\delta = \delta_2$ then $I = [z_{\delta_2}, z_0]$

Our result contrasts with the uniqueness of the oss in the absence of reforestation costs. For the sake of completeness, we briefly summarize the characterization of the oss when $\delta = 0$:

- $I(0) = \{0\}$ if, and only if, $\Delta(0) \leq 0$ (Piazza and Roy 2015).
- $I(0) = \{1\}$ if, and only if, $\Delta(1) \geq 0$ (Salo and Tahvonen 2004).
- In any other case, there exists a unique $z \in (0, 1)$ such that $\Delta(z) = 0$. $\Delta(z) = 0$ if, and only if, $I(0) = \{z\}$ (Salo and Tahvonen 2004).

It is important to note that if $\Delta(0) > 0$, the totally deforested state is not an oss when reforestation cost is zero but it is an oss if $\delta \geq \Delta(0)$.

Proposition 1 and Corollary 3 allow us to arrive at the following characterization of sustainable forest cover in an oss.

Corollary 4 *1. An oss with positive forest cover exists if, and only if, $\Delta(0) > 0$.*
2. Every oss has positive forest cover if, and only if, $\Delta(0) > \delta$. Equivalently, the total deforestation state, $\boldsymbol{x}(0)$, is an oss if, and only if, $\Delta(0) \leq \delta$.
3. A steady state $\boldsymbol{x}(1)$ with full forest cover is an oss if, and only if, $\Delta(1) \geq 0$. It is the unique oss if, and only if, $\Delta(1) \geq \delta$.

The broad intuition behind the corollary above is the following. The reforestation cost acts as a barrier to moving to larger forest covers but should not affect a transition in the other direction. Even if increasing the forest cover yields a larger benefit, it may be outweighed by the costs of reforestation. Roughly speaking, a state $\boldsymbol{x}(z)$ might be an oss for a large enough δ, even if z is too small to make $\boldsymbol{x}(z)$ an oss in the absence of reforestation costs. However, if reducing the forest cover is optimal, change in reforestation cost does not change that fact, i.e., a state $\boldsymbol{x}(z)$ where the forest cover z is too large cannot be transformed into an oss by increasing δ.

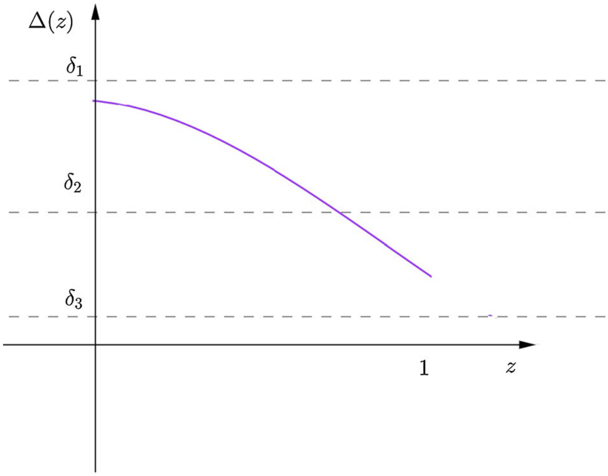

Fig. 3 If $\delta = \delta_1$ then $I = [0, 1]$, if $\delta = \delta_2$ then $I = [z_{\delta_2}, 1]$ and if $\delta = \delta_3$ then $I = \{1\}$

3.1 Comparative statics

It is easy to see that high timber price ($u'(\cdot)$) will favor oss with larger forest covers. Analogously, high marginal return on the alternative use ($w'(\cdot)$) is likely to favor oss with smaller forest covers. In addition, if $u'(0) = \infty$ then total deforestation is never an oss as $\Delta(0) > \delta$.

The effect of a variation of the discount factor b is more difficult to characterize as the value of σ depends on b. We can prove that higher values of b favor oss with larger forest cover in segments where the value of σ does not change. Indeed, we will see that $\Delta(z)$ is increasing with b (and so are z_0 and z_δ) in the subintervals where σ remains constant. To this end, we state $\Delta(z)$ as

$$\Delta(z) = \frac{b^\sigma}{1 - b^\sigma} \left(f_\sigma u'(f_\sigma z) - \frac{b}{1-b} \frac{1 - b^\sigma}{b^\sigma} w'(1 - z) \right),$$

and observe that the term $\frac{b^\sigma}{1-b^\sigma}$ is increasing with $b \in (0, 1)$ and the term $\frac{b}{1-b}\frac{1-b^\sigma}{b^\sigma} = \sum_{i=0}^{\sigma-1} b^{-i}$ is decreasing with $b \in (0, 1)$.[18]

We characterize now the limiting behavior of $I(\delta)$ when b goes to one.[19] To simplify the study, we take b close enough to one such that σ does not change when $b \to 1$,

[18] In the special case of a single-age forest, i.e., $n = 1$, σ does not change with other parameters and we can readily conclude that z_0 and z_δ increase monotonically with b. When $n > 1$, the age structure of the forest complicates the comparative statics.

[19] We thank an anonymous referee for suggesting this characterization.

$$\lim_{b\to1}\Delta(z)=\lim_{b\to1}\frac{1}{1-b}\left[\frac{1-b}{1-b^\sigma}f_\sigma b^\sigma u'(f_\sigma z)-bw'(1-z)\right]$$
$$=\left[\frac{f_\sigma}{\sigma}u'(f_\sigma z)-w'(1-z)\right]\lim_{b\to1}\frac{1}{1-b}.$$

Evidently, $\lim_{b\to1}\frac{1}{1-b}=+\infty$. If z is such that the term between brackets is positive, then $\lim_{b\to1}\Delta(z)=+\infty$ and there is no oss with this level of forest cover. (The forest cover is too small to be sustained as an oss as discounting vanishes.) If z is such that the term between brackets is negative, then $\lim_{b\to1}\Delta(z)=-\infty$ and once again, there is no oss with this level forest cover. (The forest cover is too large to be sustained as an oss as discounting vanishes.) Observing that, as the forest cover increases, the term between brackets decreases, we see that there are three different scenarios.

First, if $\lim_{b\to1}(1-b)\Delta(0)=\frac{f_\sigma}{\sigma}u'(0)-w'(1)\le0$, then $\lim_{b\to1}\Delta(z)=-\infty$ for every $z\in[0,1]$ and the set of oss converges to the fully deforested steady state $\boldsymbol{x}(0)$ as discounting vanishes. Second, if $\lim_{b\to1}(1-b)\Delta(1)=\frac{f_\sigma}{\sigma}u'(f_\sigma)-w'(0)\ge0$ then $\lim_{b\to1}\Delta(z)=\infty$ for every $z\in[0,1]$ and the set of oss converges to a steady state $\boldsymbol{x}(1)$ with full forest cover.

Finally, if neither of these conditions hold, then there is $\hat{z}\in(0,1)$ such that

$$\frac{f_\sigma}{\sigma}u'\left(f_\sigma\hat{z}\right)-w'(1-\hat{z})=0,\quad\text{and, in consequence}\quad\lim_{b\to1}\Delta(z)=\begin{cases}+\infty\text{ if }z<\hat{z}\\-\infty\text{ if }z>\hat{z}.\end{cases}$$

As discounting vanishes, the set of oss converges to a unique (and interior) steady state with partial forest cover $\hat{z}$.

More generally, independent of the extent of irreversibility δ, the set of optimal steady states $I(\delta)$ always converges to a singleton as discounting vanishes.[20]

3.2 Numerical example

We outline below an example that we will use to illustrate our results throughout the paper. In this section, we use the example to illustrate the characterization of the set of oss in Proposition 1 and Corollary 3 and its variation with respect to the level of irreversibility.

Consider a dual-aged forest ($n=2$) where the biomass coefficients are $f_1=0.45$ and $f_2=1$. We choose a discount factor $b=0.9$ which implies that $\sigma=2$. We assume that the benefit functions u and w are quadratic,

- $u(c)=\alpha_1c-\frac{\alpha_2}{2}c^2$ and
- $w(y)=\beta_1y-\frac{\beta_2}{2}y^2$.

In Table 1, we specify the set $I(\delta)$ for different values of the parameters α_2, β_1 and β_2 for fixed values $\delta=0.5$ and $\alpha_1=2.6$. In cases (i) and (vi) of the table, the set $I(\delta)$

[20] This suggests that in the undiscounted version of our model, the optimal steady state is probably unique, and therefore, the stability results in Brock (1970) may hold. This, however, requires further analysis given that the forestry model does not directly satisfy the assumptions of "free disposal" and "inaction" in Brock (1970).

Table 1 Interval $I(\delta)$

	α_1	α_2	β_1	β_2	I	$\Delta(0)$	$\Delta(1)$	Proposition 1
(i)	2.6	0.7	1.3	0	$\{0\}$	$-0.62 < 0$	$-2.1 < 0$	1.
(ii)	2.6	0.7	1.3	0.1	$[0, 0.12]$	$0.28 \in [0, \delta]$	$-2.1 < 0$	3.
(iii)	2.6	0.1	1.2	0	$[0, 1]$	$0.28 \in [0, \delta]$	$0.071 \in [0, \delta]$	3.
(iv)	2.6	0.7	1.3	0.3	$[0.38, 0.50]$	$2.1 > \delta$	$-2.1 < 0$	3.
(v)	2.6	0.1	1.2	0.3	$[0.85, 1]$	$3.0 > \delta$	$0.071 \in [0, \delta]$	3.
(vi)	2.6	0.7	1	0.3	$\{1\}$	$4.8 > \delta$	$0.6 > \delta$	2.

Table 2 Interval $I(\delta)$ for different values of δ

	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
(i)	$\{0\}$	$\{0\}$	$\{0\}$
(ii)	$[0.014, 0.12]$	$[0, 0.12]$	$[0, 0.12]$
(iii)	$[0.16, 1]$	$[0, 1]$	$[0, 1]$
(iv)	$[0.44, 0.50]$	$[0.38, 0.50]$	$[0.26, 0.50]$
(v)	$[0.94, 1]$	$[0.85, 1]$	$[0.68, 1]$
(vi)	$\{1\}$	$\{1\}$	$[0.90, 1]$

is a singleton: the total deforestation state at the former or the total forest cover steady state at the latter. The other four cases correspond to the existence of a continuum of steady states, i.e., the set $I(\delta)$ is an interval that may or not include each one of the extremes of $[0, 1]$. We observe that the values of $\Delta(0)$ and $\Delta(1)$ tend to increase when going from (i) to (vi).

We illustrate the dependence of $I(\delta)$ with respect to δ in Table 2

Observe that larger values of δ increases the size of $I(\delta)$ by including new oss with smaller forest covers but no modification in the oss with larger forest cover.

4 Strong forest conservation

We use the term "conservation" to broadly refer to preservation of positive forest cover. In our framework, forest land can be diverted to alternative use and this process of deforestation can reduce the forest cover to zero. In this section, we characterize a strong form of conservation where a strictly positive minimal forest cover is maintained over all time periods. The next section discusses a weaker form of conservation.

We are interested in characterizing the dynamic efficiency of conservation, i.e., the nature of economic fundamentals under which it is optimal to conserve a forest. To this end, we define the concept of global strong conservation.

Definition 1 *Global strong conservation* is said to occur if for any initial state with strictly positive forest cover, the forest cover along every optimal path is bounded away from zero, i.e., for any initial state $\boldsymbol{x}_0 = (x_{1,0}, \ldots, x_{n,0})$ where $z_o = \sum_{a=1}^{n} x_{a,0} > 0$ and an optimal program $\{\boldsymbol{x}_t\}$ from $\boldsymbol{x}_0$, there exists $\widetilde{z} > 0$ such that

$$z_t = \sum_{a=1}^{n} x_{a,t} \geq \widetilde{z} \text{ for all } t \geq 1.$$

The next proposition contains the main result of this section; it outlines a sufficient condition for global strong conservation:

Proposition 2 *There is global strong conservation if* $0 < \Delta(0) \leq \delta$

The proposition has a powerful implication: whenever $\Delta(0) > 0$, strong conservation is globally optimal if the marginal reforestation cost is large enough. In this sense, strong irreversibility in deforestation ensures forest conservation. Whether the condition $0 < \Delta(0) \leq \delta$ is *necessary* for global strong conservation remains an open question.

It is tempting to conjecture that the condition in Proposition 2 ensures conservation by making it optimal to *increase* forest cover when the area under forest is sufficiently small; that, however, is not true. The inequality $\Delta(0) \leq \delta$ in the condition in Proposition 2 implies that starting from a steady state with very small forest cover, it is not optimal to expand forest cover as the marginal reforestation cost exceeds its marginal benefit. In a series of steps outlined below, we show that $\Delta(0) \leq \delta$ has a stronger implication: reforestation is never optimal regardless of whether the current state is a potential steady state (i.e., whether or not the forest has a balanced age structure), and therefore, the area under forest cover along any optimal path is non-increasing over time. We show that this, in turn, creates a disincentive to excessively reduce the area under forest cover.

In the rest of this section, we develop the key results and arguments that lead to Proposition 2 and in this process, discover some other interesting implications. In the next subsection, we define an auxiliary function useful for characterizing uniform lower bounds on the forest cover.

4.1 Some necessary definitions

In Sect. 3, we have seen that the function $\Delta(z)$ plays a crucial role in determining whether or not there is an oss with forest cover z. To provide some characterization of the forest cover along non-stationary optimal paths, we introduce the function $g(z)$,

$$g(z) = \frac{b^\sigma f_\sigma}{1 - b^\sigma} u'(f_m z) - \frac{b}{1 - b} w'(1 - z).$$

where $f_m = \max\{f_a, a = 1, \ldots, n\}$. An interpretation of $g(z)$ is the following. Consider any path whose forest cover is below z during σ consecutive stages. Perturb this path by transferring more land to forestry at the beginning of this period and harvesting the additional forest area only after σ periods. Then, $(1 - b^\sigma)g(z)$ is a *lower bound* on the marginal benefit from this specific reforestation strategy (ignoring reforestation cost) and this bound is independent of the actual path.[21] Whenever $g(z) >$

[21] By using $f_m z$ as an upper bound of the harvest at the σ-th period, we obtain a bound that is valid for every possible path.

Table 3 Values of $\underline{z}$

	I	$\underline{z}$
(i)	$\{0\}$	0
(ii)	$[0, 0.12]$	0.073
(iii)	$[0, 1]$	$\frac{2}{3}$
(iv)	$[0.38, 0.50]$	0.37
(v)	$[0.85, 1]$	0.96
(vi)	$\{1\}$	0.84

0, the marginal benefit of dedicating more land to forestry is strictly positive for all such paths, and hence, no path where forest cover is below z during σ consecutive stages can be optimal if reforestation cost is zero. If, however, the intrinsic marginal reforestation cost $\delta > 0$, it may not be optimal to increase the forest cover from a state with forest cover z even if $g(z) > 0$.

Note that $g(z)$ is decreasing. Let $\underline{z} \in [0, 1]$ be defined by

$$\underline{z} = \sup\{z \in [0, 1] : g(z) \geq 0\}.$$

We define:

$$\underline{z} = 0, \text{ if } g(0) \leq 0.$$

Observe that $g(0) = \Delta(0)$ so that $\underline{z} > 0$ if $\Delta(0) > 0$. Also, if $g(1) \leq 0$ and $g(0) \geq 0$ then $g(\underline{z}) = 0$.

Table 3 presents the values of $\underline{z}$ for the various cases of the example outlined in Sect. 3.2.

4.2 Minimal forest cover

We are now ready to state an important lemma:

Lemma 4 *Suppose that $\Delta(0) \leq \delta$. Let $\{x_t\}$ be an optimal path and let $\{z_t\}$ be the path of area under forest cover associated with it. Then, $z_t \geq z_{t+1}$ for all t, i.e., the area under forest cover is always non-increasing along an optimal path. If, further, $\Delta(0) > 0$, then*

$$z_t \geq \min\{\underline{z}, z_o\} \text{ for all } t.$$

Lemma 4 states that if the intrinsic marginal reforestation cost δ is large enough to exceed $\Delta(0)$, then reforestation is never optimal and the area under forest cover is always non-increasing along an optimal path; this holds regardless of whether or not $\Delta(0)$ is positive. An immediate implication of this is that any increase in the irreversibility parameter δ beyond $\Delta(0)$ has no further effect on the optimal path (and is therefore identical to full irreversibility). This proof relies on showing that $\Delta(0)$

is always an upper bound on the marginal benefit from expansion of forest area. The condition $\delta \geq \Delta(0)$ on the reforestation case will be called *strong irreversibility*.

Lemma 4 then states that if, in addition to strong irreversibility, $\Delta(0) > 0$, then the area under forest cover along an optimal path is bounded below by $\min\{\underline{z}, z_o\}$. Although the condition for this lower bound on the area under forest cover requires strong irreversibility ($\delta \geq \Delta(0)$), the lower bound itself is independent of the extent of irreversibility δ.

Note that $\Delta(0) = g(0)$ and, hence, $\Delta(0) > 0$ implies that $g(0) > 0$, and therefore, $\underline{z} > 0$. Lemma 4 therefore implies that when $0 < \Delta(0) \leq \delta$, the area under forest cover is bounded away from zero along every optimal path if the initial forest cover $z_o > 0$; in other words, we have global strong conservation and Proposition 2 holds.

Further, as long as the initial forest cover z_o exceeds the critical level $\underline{z} > 0$, the forest cover along all optimal programs is uniformly bounded below by $\underline{z}$. The value of $\underline{z}$ depends on the marginal benefits of the two productive activities (forests and alternative use). The critical forest size $\underline{z}$ is likely to be larger if marginal benefit from harvesting timber is higher relative to that from alternative use; hence, it depends on the curvature of u and w, i.e., how these marginal benefits decline as more timber is harvested or as more land is diverted to alternative use. The effect of a variation of the discount factor b in the value of $\underline{z}$ is very similar to the effect such a variation has on z_0 and z_δ, which was studied in Sect. 3.1. This is so because the functions $\Delta(z)$ and $g(z)$ have a similar structure. We can affirm then, that higher values of b will induce higher levels of the critical level $\underline{z}$.

To understand why the conditions in Lemma 4 imply that the area under forest on the optimal path is bounded below by $\min\{\underline{z}, z_o\}$, suppose to the contrary that there is an optimal path where in a certain period the forest area is reduced (for the first time) to a level $z < \min\{\underline{z}, z_o\}$. By definition, $g(z) > 0$. Now consider a variation of this optimal path where instead of reducing the forest area all the way to z, it is reduced to a level $z + \epsilon$ where $\epsilon > 0$ is arbitrarily small; the additional forest area gained through this variation is now clear cut and allowed to grow for σ periods and then harvested (and consumed). Note that this variation involves no reforestation cost. Further, the fact that the forest area along the optimal path is non-increasing over time ensures that the suggested variation is always feasible (holding the additional ϵ forest area for σ periods only reduces the area under alternative use by ϵ in those periods and requires no other modification to the optimal program). It is easy to check that the net benefit from this variation is identical to that in the case where reforestation cost is zero and starting from forest area z one expands the forest area to $z+\epsilon$, harvesting the additional forest area only after σ periods. As discussed in the previous subsection, the marginal gain in the latter case is bounded below by $(1-b^\sigma)g(z) > 0$. Therefore, regardless of the reforestation cost, the net marginal effect of the proposed variation to the optimal path is strictly positive. Hence, it cannot be optimal to reduce the forest area to z.

The condition $0 < \Delta(0) \leq \delta$ is represented in Figs. 2 and 3 when $\delta = \delta_1$. Among the cases presented in Table 3, only (ii) and (iii) satisfy the condition $0 < \Delta(0) \leq \delta$. It is worth noting from Proposition 1 that condition $0 < \Delta(0) \leq \delta$ is also necessary and sufficient for having a continuum of optimal steady states including one with zero forest cover state.

As mentioned above, Lemma 4 implies that if $0 < \Delta(0) \leq \delta$ and the initial area under forest cover $z_o \geq \underline{z}$, then the forested area along the optimal path always lies above $\underline{z}$. However, the lemma also indicates that the area under forest cover is non-increasing along every optimal path, and therefore, the area under forest cover in this case always lies in the interval $[\underline{z}, z_o]$. If, however, $z_o \leq \underline{z}$ then the area under forest cover always lies above z_o; but as the area under forest cover never increases strictly, the forest size is constant over time. (This does not mean it is a steady state as the age structure of the forest can change over time.) We summarize this in the following useful proposition:

Proposition 3 *Assume* $0 < \Delta(0) \leq \delta$.

1. *If the initial area under forest cover is larger than the critical level* $\underline{z}$*, then the area under forest cover along the optimal path is non-increasing in size but never falls below this critical level. In particular, the area under forest cover always remains between* $\underline{z}$ *and* z_o.
2. *If the initial area under forest cover is smaller than the critical level* $\underline{z}$*, then the area under forest cover remains unchanged over time, i.e., the forest is managed as if it were a pure forest of total area* z_o *with no alternative use; this holds regardless of whether the initial state is an oss.*

We have not been able to generalize Propositions 2 and 3 to the case $\delta < \Delta(0)$. This is because our current proofs rely heavily on the monotonicity of the forest cover, which is assured by Lemma 4, thanks to the condition $\delta \geq \Delta(0)$. If this last condition does not hold, we expect non-monotone changes in the forest cover. This is the reason why in the two following sections we focus on lower and upper bounds of the forest cover.

5 Weak forest conservation

In this section, we focus on a weaker concept of forest conservation: the avoidance of total deforestation.

As mentioned in the introduction, the concept of deforestation is depletion of the area under forest and diversion of such land to alternative (non-forest use). Total deforestation refers to complete removal of forest cover. We define two concepts of total deforestation:

Definition 2 A program $\{\boldsymbol{x}_t\}_{t=0}^{\infty}$ is said to be characterized by <u>immediate total deforestation</u> if z_t is non-increasing and

$$z_t = 0 \text{ for all } t \geq n.$$

Definition 3 A program $\{\boldsymbol{x}_t\}_{t=0}^{\infty}$ is said to be characterized by <u>eventual total deforestation</u> if

$$\lim_{t\to\infty} z_t = 0.$$

The latter is a weaker concept of total deforestation: A path characterized by immediate total deforestation is also characterized by eventual total deforestation. Avoidance of eventual total deforestation requires that the forest cover is bounded away from zero for at least a subsequence of time periods, i.e., $\limsup z_t > 0$. This leads to the concept of *weak conservation* defined below:

Definition 4 There is *global weak conservation* if no optimal program starting from a state with strictly positive forest cover is characterized by eventual total deforestation.

Observe that like the concept of global strong conservation, the concept of global weak conservation does not impose any requirement on optimal paths from an initial state with zero forest cover. Unlike global strong conservation, global weak conservation allows for the possibility that the forest cover is reduced to zero over a subsequence of time periods, i.e., $\liminf z_t = 0$.

From Corollary 4, we know that $\Delta(0) > 0$ is a necessary and sufficient condition for there to be at least one optimal steady state with nonzero forest cover. Hence, $\Delta(0) > 0$ is sufficient to ensure that there is at least one optimal path that is not characterized by eventual deforestation. As it turns out, $\Delta(0) > 0$ ensures something much stronger; it is, in fact, exactly what is needed to rule out eventual total deforestation from any initial state with positive forest cover, i.e., to ensure global weak conservation.

Proposition 4 *There is global weak conservation if, and only if,* $\Delta(0) > 0$.

In the proof of this proposition, we show that if there is an optimal path such that the forest cover $z_t \to 0$, the necessary first-order optimality conditions imply that $\Delta(0) \leq 0$. Therefore, $\Delta(0) > 0$ is sufficient to have global weak conservation. A surprising fact is that $\Delta(0) > 0$ is also a necessary condition for the existence of at least one path along which total deforestation, either eventual or immediate, does not occur. Prior work has shown that if $\Delta(0) \leq 0$, then immediate total deforestation is always optimal when there is no reforestation cost (Piazza and Roy 2015). As z_t is non-increasing along such a path with immediate deforestation, introducing reforestation cost does not affect the immediate returns from such a path and so this path continues to be optimal when reforestation cost is positive.

Proposition 4 can be extended to,

Proposition 5 *The following are equivalent*

1. *There is global weak conservation*
2. $\Delta(0) > 0$
3. *There exists at least one oss with strictly positive forest cover*
4. *There exists at least one optimal path where eventual total deforestation does not occur*
5. *There exists at least one optimal path where immediate total deforestation does not occur*

In the previous section, we have seen that $0 < \Delta(0) \leq \delta$ is sufficient to have global strong conservation. The above results indicate that even if irreversibility is weak, i.e., $\Delta(0) \geq \delta$, global weak conservation remains optimal as long as $\Delta(0) > 0$. Indeed, Proposition 4 implies that

Corollary 5 *The extent of irreversibility or reforestation cost (δ) has no effect on global weak conservation.*

However, the value of δ does affect the optimal path starting from the one initial state not considered in the definition of global weak conservation: the zero forest cover state. We now briefly discuss the effect of δ on the possibility of escaping from the initial state that is totally deforested.

From Corollary 2, we know that $\boldsymbol{x}(0)$ is an oss if, and only if, $\Delta(0) \leq \delta$; thus, if the level of irreversibility is large enough the forest cannot be optimally regenerated from zero forest cover. On the other hand, if $\delta < \Delta(0)$ then $\boldsymbol{x}(0)$ is not an oss so that starting from zero forest cover, the next period's forest cover $z_1 > 0$.[22] We can now apply Proposition 4 from stage $t = 1$ to infer that the forest cover does not converge to zero, i.e., total deforestation is avoided. In this sense, stronger irreversibility works against regeneration of forests from a totally deforested state.

A value of $0 \leq \delta < \Delta(0)$ in principle allows the forest area to expand. As we just mentioned, total deforestation state is no longer an oss, and some degree of reforestation is optimal when starting from a totally depleted area. However, the fact that reforestation costs no longer prevent reforestation along every optimal path introduces non-monotonicity and technical difficulties that prevent us from establishing uniform lower bounds on the forest cover. This is in contrast to the case of strong irreversibility $\delta \geq \Delta(0)$ where, as we have shown in the previous section (see Lemma 4), the optimal forest area is monotone over time and one can derive a uniform lower bound $\min\{\underline{z}, z_o\}$ on the forest cover along an optimal path. However, with weak irreversibility, $\min\{\underline{z}, z_o\}$ is still a level that the forest cover along an optimal path must exceed from time to time (and in fact, at least once every σ time periods):

Proposition 6 *If $\{x_t\}$ is an optimal program and $\{z_t\}$ is its associated path of area under forest cover, then for every $t \geq 0$,*

$$\max_{j=1,\ldots,\sigma} \{z_{t+j}\} \geq \min\{\underline{z}, z_o\} \tag{7}$$

Recall that $\underline{z}$ is the lower bound of forest covers z for which $g(z) > 0$ and $\underline{z} > 0$ if $\Delta(0) = g(0) > 0$. The key argument behind Proposition 6 is similar to that in Lemma 4. If one reduces the forest cover to a level below z and remains there for σ periods, then one can do a variation where the forest cover is reduced to a level that is slightly higher and the additional forest is harvested only after σ periods; the marginal net gain from this variation taking into account consumption after σ periods and loss of alternative use for these periods is bounded below by $(1 - b^\sigma)g(z)$ as the forest cover on the initial path remains below z for σ periods. If $g(z) > 0$, the variation is gainful and so the initial path cannot be optimal. Thus, the forest cover must exceed z at least once every σ periods.

[22] Having $z_o = 0$ and $z_1 = 0$ implies that the zero forest cover is an oss.

Table 4 Values of $\bar{z}(q)$

	I	$\bar{z}(0)$	$\bar{z}(\delta)$
(i)	$\{0\}$	0	0
(ii)	$[0, 0.12]$	0.32	0
(iii)	$[0, 1]$	1	1
(iv)	$[0.38, 0.50]$	0.77	0.59
(v)	$[0.85, 1]$	1	0.92
(vi)	$\{1\}$	1	1

6 Upper bounds on forest cover

In this section, we outline some upper bounds on optimal forest cover that provide us with some understanding of limits on the efficiency of forest conservation.

In Sect. 4, we defined the function $g(z)$ that is a lower bound on the marginal benefit of dedicating more land to forestry and following a certain policy in the absence of reforestation costs. We now define a function h that provides an upper bound on this marginal benefit:

$$h(z) = \frac{b^\sigma f_\sigma}{1 - b^\sigma} u'(0) - \frac{b}{1 - b} w'(1 - z).$$

It is easy to check that $h(z)$ is continuous and non-increasing.

Given any path where the forest cover is always above z, consider the possibility of allocating more land to the alternative use without bringing it back to forestry. We show that $-h(z)$ is a lower bound on the net marginal benefit of such reallocation. So, if $h(z) < 0$, it is optimal to take land out of the forestry for alternative use. Hence, if the forest cover is always above z a necessary condition for the path to be optimal is that $h(z) \geq 0$. The flip side of this is that for any path where the forest cover is always above z (and the current forest cover is less than 1), the net marginal benefit from expanding forest cover (which involves reforestation cost) is bounded below by $h(z) - \delta$. Thus, a necessary condition for this kind of path to be optimal is that $h(z) - \delta \leq 0$. We are therefore interested in knowing when $h(z) \geq 0$ and when $h(z) \geq \delta$. With this in mind we define, for $q = 0, \delta$

$$\bar{z}(q) = \sup \{z \in [0, 1] : h(z) \geq q\} \tag{8}$$

and $\bar{z}(q) = 0$ if $h(0) < q$. Recall that $\underline{z}$ is the lower bound of forest covers z for which $g(z) > 0$. It is easy to check that $\underline{z} \geq \bar{z}(0)$ with equality if, and only if, $\underline{z} = 0$. Finally, as $h(0) = \Delta(0)$ we have that $\bar{z}(q) > 0$ if, and only if, $\Delta(0) > q$.

For the example in Sect. 3.1, we compute the values for $\bar{z}(q)$ (see Table 4).

Observe that the curvature of u does not affect the value of $\bar{z}(q)$. The variation of $\bar{z}(q)$ with respect to the other primitives of the model is analogous to that of $\underline{z}$, i.e., increasing with the discount factor b and decreasing with the marginal benefit of the alternative use.

We begin with a result which indicates that forest covers above a certain level cannot be uniformly sustained in the long run:

Proposition 7 *If* $\{x_t\}$ *is an optimal program and* $\{z_t\}$ *is its associated path of area under forest cover, then*

$$\liminf_t z_t < \bar{z}(0).$$

Proposition 7 implies that the (long-run) forest cover must fall below $\bar{z}(0)$ infinitely often. Under strong irreversibility, i.e., $\delta \geq \Delta(0)$ we know from Lemma 4 that there is no reforestation along the optimal path and forest cover is monotone and hence, convergent over time; in that case, Proposition 7 can be strengthened to derive an upper bound on the long-run forest cover:

Corollary 6 *Suppose* $\delta \geq \Delta(0)$. *If* $\{x_t\}$ *is an optimal program and* $\{z_t\}$ *is the path of area under forest cover associated with this optimal program, then* $\lim_t z_t < \bar{z}(0)$.

Note that if $\Delta(0) \leq 0$, then $\bar{z}(0) = 0$ and the above result implies that eventual total deforestation is optimal; this is consistent with Proposition 4.

If $\Delta(0) > \delta$ (weak irreversibility), then the forest cover may increase along the optimal path. The next proposition outlines an upper bound in the latter case; the forest cover must fall below this upper bound from time to time (and not just in the long run).

Proposition 8 *If* $\{x_t\}$ *is an optimal program,* $\{z_t\}$ *is its associated path of area under forest cover and there is* T *such that* $z_T < z_{T+1}$ *(forest cover increases in period* T*) then*

$$\min_{j=1,\ldots,n} \{z_{t+j}\} \leq \bar{z}(\delta) \textit{ for all } t \geq T. \tag{9}$$

Proposition 8 implies that either z_t is non-increasing for all t or there exists T such that $\min_{j=1,\ldots,n}\{z_{t+j}\} \leq \bar{z}(\delta)$ for all $t \geq T$. It follows that:

Corollary 7 *If* $\{x_t\}$ *is an optimal program and* $\{z_t\}$ *is the path of area under forest cover associated with this optimal program, then*

$$\min_{j=1,\ldots,n} \{z_{t+j}\} \leq \max\{z_o, \bar{z}(\delta)\}$$

As the function h is decreasing, the bound $\bar{z}(\delta)$ is strictly decreasing with δ. That means that the upper bound on the forest cover is decreasing with the level of irreversibility or the marginal reforestation cost. In this sense, stronger irreversibility makes it more difficult to sustain larger forest covers. This is qualitatively aligned with the conclusion from our analysis of optimal steady states where stronger irreversibility only adds steady states with smaller forest cover. However, as we have seen in Sect. 4, strong irreversibility also helps ensure a strictly positive minimal forest cover.

Note that $\bar{z}(\delta) < \bar{z}(0)$ meaning that the bound provided by Proposition 8 is stricter than the one of Proposition 7. Indeed,

$$\min_{j=1,\ldots,n} \{z_{t+j}\} \leq \bar{z}(\delta) \text{ for all } t \geq T \Rightarrow \liminf_{t} z_t < \bar{z}(\delta) < \bar{z}(0).$$

However, Proposition 8 can only be applied to optimal paths along which there is reforestation at least once, something that we do not know directly from the primitives of the model, whereas Proposition 7 can be applied more generally.

7 Single-age forest

In this section, we consider the special case where $n = 1$ so that trees live for only one period and the forest is fully harvested every period. In this case, there is no forest rotation and as we shall see, optimal forest size exhibits monotone dynamics (independent of the extent of irreversibility). The latter allows us to obtain a pretty sharp characterization of conservation and the effects of irreversibility.

Let us first state the dynamic optimization problem in this case:

$$\begin{cases} \text{maximize} & \sum_{t=0}^{\infty} b^t [u(f_1 z_t) + w(1 - z_t) - v(z_{t+1} - z_t)] \\ \text{subject to} & z_t \in [0, 1] \end{cases} \tag{10}$$

where $x_{1,t} = z_t$ and $C_t = x_{1,t}$. (The last equality comes from the fact that all the trees die at the end of the period, so it is not optimal to leave them unharvested.)

In the absence of irreversibility, the problem is essentially a static problem where in every period, and independent of history, one chooses z in $[0, 1]$ so as to maximize

$$\{u(f_1 z) + w(1 - z)\},$$

which generates a forest of constant size z^s in every period (after the initial period), where z^s is the (unique) solution to this static maximization problem. One can check that in fact, z^s is identical to the forest cover in the optimal steady state with the largest forest as defined in Sect. 3 for the general case. However, when reforestation costs are introduced, the optimal paths exhibit a more complex behavior,

Lemma 5 *The solution to* (10), $\{z_t\}$, *satisfies the following:*

1. *If the initial forest cover z_o is larger or equal to z^s, the static optimal forest size, then the optimal path is one where $z_t = z^s$ for all $t \geq 1$.*
2. *If the initial forest cover z_o is below z^s, the optimal path is one where $\{z_t\}$ is non-decreasing, bounded above by z^s and converges to a value in the interval $[z_o, z^s]$*

The above lemma indicates that z^s is a critical point. We are now ready to state the main result of this section.

Proposition 9 *Assume $n = 1$, then the following are equivalent*

1. *There is global strong conservation*
2. *There is global weak conservation*
3. $\Delta(0) > 0$.

Observe that the above result strengthens Proposition 2 (for the case where $n = 1$). It indicates that for this special case, the extent of irreversibility does not play any role in forest conservation. This raises the question about whether irreversibility can affect the extent of forest cover.

Suppose that the condition for global conservation is satisfied, i.e., $\Delta(0) > 0$. For large initial forests that are above the static optimal forest size z^s, irreversibility has no effect as the forest transitions to the static optimal size in one period and remains there. For the range of initial forests that are smaller than the static optimal level, the optimal path is either increasing or constant; an increase in irreversibility simply reduces the threshold below which it is optimal to expand forests[23], i.e., it expands the set of optimal steady states by adding states with smaller forest cover.

8 Numerical example

In the previous two sections, we have derived certain analytical bounds on the forest cover along an optimal path. To understand the nature of these bounds, how they vary with specific parameters and what they imply for the optimal path we return to the parametric example in Sect. 3.1. This illustrates the usefulness of our results.

To begin with, we specify the values of the bounds for the six cases of parameter values listed in Table 1.

	I	$\underline{z}$	$\bar{z}(0)$	$\bar{z}(\delta)$	Comment
(i)	{0}	0	0		$\Delta(0) \leq 0$
(ii)	[0, 0.12]	0.073	0.32		$\Delta(0) \in (0, \delta)$
(iii)	[0, 1]	$\frac{2}{3}$	1		$\Delta(0) \in (0, \delta)$
(iv)	[0.38, 0.50]	0.37	0.77	0.59	$\Delta(0) > \delta$
(v)	[0.85, 1]	0.95	1	0.92	$\Delta(0) > \delta$
(vi)	{1}	0.84	1	1	$\Delta(0) > \delta$

(i) In the first case, the condition $\Delta(0) \leq 0$ assures that there is immediate deforestation starting from any initial conditions, and, of course, the total deforestation state is the unique oss.

(ii) In the second case, the fact that $\Delta(0) \leq \delta$ assures that there is strong irreversibility of deforestation and that $I(\delta)$ is a closed interval including $x = 0$.

[23] This threshold is, in fact, identical to z_δ, i.e., the optimal steady state with smallest forest cover as defined in Sect. 3. Indeed, any state in the interval $[z_\delta, z^s]$ is a steady state, and hence, the optimal path is constant. If $z_o < z_\delta$, then the optimal path must be strictly increasing and converging to a steady state.

Furthermore, computing $\underline{z}$ and $\bar{z}(0)$ we see that optimal paths can be classified into three cases according to the initial forest cover:

- If $z_o \leq \underline{z} = 0.073$, the forest will be managed as a pure forest of area z_o.
- If $\underline{z} < z_o < \bar{z}(0) = 0.32$, the forest cover will be non-increasing, but will never go below the lower bound $\underline{z}$; hence, its limit will be some value between $\underline{z}$ and z_o.
- If $z_o \geq \bar{z}(0)$, the forest cover will decrease monotonically to some level between $\underline{z}$ and $\bar{z}(0)$

(iii) In this case, every area of land can support an optimal steady state. The value of $\underline{z} = \frac{2}{3}$ implies that any forest below that level will be managed as a pure forest and that larger forests will remain above it. Nothing more can be said as the upper bounds found for this case are equal to 1, the maximum possible area.

(iv) This is the first case of weak irreversibility where $\Delta(0) > \delta$; here, $\boldsymbol{x}(0)$ is no longer an oss. We observe that interval of forest covers that are sustained by optimal steady states is quite small.
If the path of optimal forest cover happens to be non-increasing, then it is either always equal to z_o or above $\underline{z} = 0.37$. Furthermore, in the long run it must be below $\bar{z}(0) = 0.77$ for every possible initial level z_o.
If the path is non-monotone, the conclusions that can be drawn are weaker:

- If $z_o < \underline{z} = 0.37$ then $z_o \leq \max_{j=1,2}\{z_{t+j}\}$ and $\min_{j=1,2}\{z_{t+j}\} \leq \bar{z}(\delta) = 0.59$
- If $\underline{z} \leq z_o \leq \bar{z}(\delta)$ then $\underline{z} \leq \max_{j=1,2}\{z_{t+j}\}$ and $\min_{j=1,2}\{z_{t+j}\} \leq \bar{z}(\delta)$
- If $z_o > \bar{z}(\delta)$ then $z_o \leq \max_{j=1,2}\{z_{t+j}\}$ and $\min_{j=1,2}\{z_{t+j}\} \leq z_o$.

In other words, a necessary (but not sufficient) condition for the forest area to remain uniformly above $\bar{z}(\delta)$ is to start above it. Analogously, to stay constantly below $\underline{z}$ it is necessary to have the initial forest area z_o lie below it.
The following two cases are very similar to (iv), except that we have no information on upper bounds.

(v) For any optimal program where the area under forest is non-increasing, the forested area will remain either equal to z_o or above $\underline{z} = 0.95$. In the general case, we can say:

- If $z_o < \bar{z}(\delta) = 0.92$ then $z_o \leq \max_{j=1,2}\{z_{t+j}\}$ and $\min_{j=1,2}\{z_{t+j}\} \leq \bar{z}(\delta) = 0.95$
- If $\bar{z}(\delta) \leq z_o \leq \underline{z}$ then $z_o \leq \max_{j=1,2}\{z_{t+j}\}$ and $\min_{j=1,2}\{z_{t+j}\} \leq z_o$
- If $z_o > \underline{z}$ then $\underline{z} \leq \max_{j=1,2}\{z_{t+j}\}$ and $\min_{j=1,2}\{z_{t+j}\} \leq z_o$.

(vi) If the optimal path of forest cover is non-increasing, it will remain either equal to z_o or above $\underline{z} = 0.84$. For non-monotone path, we only know that $\max_{j=1,2}\{z_{t+j}\} \geq \min\{z_o, \underline{z}\}$.

Appendix A: Results without strict concavity of $u(\boldsymbol{x})$

In this section, we briefly outline modified versions of some of our results that hold when A_2 is replaced by the a weaker assumption on u:

A_2': u is concave in $\mathbb{R}_+$.[24]

In the absence of strict concavity of both u and w, we can no longer assert uniqueness of optimal program. The possibility of having non-unique optimal programs requires us to modify almost all the definitions and statements of this paper. Instead of doing this, we will illustrate the role of strict concavity for the monotonicity and convergence of the optimal path by continuing to assume that there is a unique optimal path from every initial state. Under this assumption, we describe how the results of Sects. 3 and 5 are modified when A_2 is replaced by A_2'. (Results in other sections are not substantively modified.)

Alternative results of Sect. 3

In the absence of A_2, the definitions of z_0 and z_δ have to be adapted to the fact that the equations $\Delta(x) = 0$ and $\Delta(x) = \delta$ may not define a unique solution:

$$z_0 = \min\{z \in [0, 1] : \Delta(z) = 0\}$$
$$z_\delta = \max\{z \in [0, 1] : \Delta(z) = \delta\}$$

Proposition 1 has a weaker version where 1. and 2. must be modified.

Proposition 1′ *Under A_1, A_2', A_3 and A_4, the set of oss coincides with $\{\boldsymbol{x}(z)/z \in [z_\delta, z_0]\}$, i.e., the set of states with forest cover $z \in [z_\delta, z_0]$ where the forest area z is evenly divided among trees of age classes 1 through σ. In particular,*

1. *If $\Delta(0) < 0$, then $I(\delta) = \{0\}$ for all $\delta \geq 0$, i.e., regardless of the extent of irreversibility (cost of reforestation) the totally deforested state is the unique oss.*
2. *If $\Delta(1) > \delta$, then $I(\delta) = \{1\}$, i.e., there is a unique oss and it is characterized by full forest cover.*
3. *If $\delta > 0$, $\Delta(0) > 0$ and $\Delta(1) < \delta$, then $z_\delta < z_0$ and there is a continuum of oss; the set of forest covers that can be sustained in an oss is a closed interval $[z_\delta, z_0]$ with non-empty interior. In addition, $I(\delta) = \{z \in [0, 1] : 0 \leq \Delta(z) \leq \delta\}$.*

And, finally, the statement of Corollary 4 has to be modified (because certain properties are no longer equivalent).

Corollary 4′
1. *An oss with positive forest cover exists if $\Delta(0) > 0$.*
2. *Every oss has positive forest cover if, and only if, $\Delta(0) > \delta$. Equivalently, the total deforestation state, $\boldsymbol{x}(0)$, is an oss if, and only if, $\Delta(0) \leq \delta$.*
3. *A steady state $\boldsymbol{x}(1)$ with full forest cover is an oss if, and only if, $\Delta(1) \geq 0$. A sufficient condition for uniqueness is $\Delta(1) > \delta$.*

[24] We want to consider not only linear functions but also the larger set of non-linear, concave functions like, for example, concave, piece-wise linear concave functions.

Alternative results of Sect. 5

Section 5 characterizes necessary and sufficient conditions to have global weak conservation. When A_2 is replaced by A_2', the necessary condition is weaker.

Proposition 4′ *Under A_1, A_2', A_3 and A_4*

$$\Delta(0) > 0 \Rightarrow \textit{There is global weak conservation} \Rightarrow \Delta(0) \geq 0$$

Analogously, the equivalence in Proposition 5 is lost. Indeed, all that can be said is that

$$(2) \Rightarrow (1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow \Delta(0) \geq 0.$$

It is worth pointing out that all the results of Sect. 5 can be retrieved in their original form under an additional assumption:

– u is strictly concave in a neighborhood of $x = 0$.

This additional assumption and A_2' are still weaker than A_2.

Appendix B: Proofs

Preliminary results: Euler conditions

Before presenting the proofs of the results stated in the main part of the paper, we state and prove some useful Euler inequalities that any optimal program must satisfy.

Lemma 6 *Let $\{\boldsymbol{x}_t\}_{t=0}^{\infty}$ be an optimal program.*
If $\min_{j=1\ldots a}\{y_{t+j}\} > 0$ for some $t \geq 0$, then

$$\sum_{j=1}^{a} b^j w'(y_{t+j}) + v'_+(z_{t+1} - z_t) \geq b^a f_a u'(c_{t+a}) + b^a v'_-(z_{t+a+1} - z_{t+a}). \quad (11)$$

If $c_{a,t+a} > 0$ for some t, then

$$\sum_{j=1}^{a} b^j w'(y_{t+j}) + v'_-(z_{t+1} - z_t) \leq b^a f_a u'(c_{t+a}) + b^a v'_+(z_{t+a+1} - z_{t+a}). \quad (12)$$

Proof We define the unitary vector $\boldsymbol{e}_j \in \mathbb{R}^n$ such that $e_j = 1$ and $e_k = 0$ for all $k \neq j$. Consider an alternative program $\{\widehat{\boldsymbol{x}}_s\}_{s=0}^{\infty}$ such that

$$\begin{aligned} \widehat{\boldsymbol{x}}_{t+j} &= \boldsymbol{x}_{t+j} + \epsilon \boldsymbol{e}_j \qquad \forall j = 1, \ldots, a. \\ \widehat{\boldsymbol{x}}_s &= \boldsymbol{x}_s \qquad\qquad \text{else} \end{aligned}$$

In words, whenever $\epsilon > 0$ ($\epsilon < 0$), the alternative program is one where the area under alternative use is reduced (increased) by ϵ at stage $t+1$ and replanted to yield more (less) young forest next period. This modification of the young forest area is allowed to grow undisturbed until age a at which point the consumption of forest of age a is modified. Of course, the area under alternative use can only be reduced if it is strictly positive along the periods involved, i.e., $\min_{j=1,\ldots,a}\{y_{t+j}\} > 0$. On the other hand, to increase the land under alternative use along the stages $t+1, \ldots, t+a$, we decrease the harvest of age class a at stage $t+a$; hence, we need $c_{a,t+a} > 0$. In conclusion, the alternative program is feasible for $-c_{a,t+a} < \epsilon < \min\{y_{t+1}, y_{t+2}, \ldots, y_{t+a}\}$.

As $\{\boldsymbol{x}_t\}_{t=0}^{\infty}$ is optimal, the modification must not be gainful, hence:

$$\sum_{j=1}^{\infty} b^{j-1}[u(c_j) + w(y_j) - v(z_{j+1} - z_j)] \geq \sum_{j=1}^{\infty} b^{j-1}[u(\widehat{c}_j) + w(\widehat{y}_j) - v(\widehat{z}_{j+1} - \widehat{z}_j)]$$

which is equivalent to

$$b^a u(c_{t+a}) + \sum_{j=1}^{a} b^j w(y_{t+j}) - v(z_{t+1} - z_t) - b^a v(z_{t+a+1} - z_{t+a}) \geq b^a u(c_{t+a} + f_a\epsilon) + \sum_{j=1}^{a} b^j w(y_{t+j} - \epsilon) - v(z_{t+1} + \epsilon - z_t) - b^a v(z_{t+a+1} - z_{t+a} - \epsilon),$$

and reordering we get

$$\sum_{j=1}^{a} b^j [w(y_{t+j}) - w(y_{t+j} - \epsilon)] - v(z_{t+1} - z_t) + v(z_{t+1} - z_t + \epsilon) \geq b^a [u(c_{t+a} + f_a\epsilon) - u(c_{t+a})] + b^a v(z_{t+a+1} - z_{t+a}) - b^a v(z_{t+a+1} - z_{t+a} - \epsilon)$$

If $\min_{j=1\ldots a}\{y_{t+j}\} > 0$ we can consider $\epsilon > 0$. Dividing through by ϵ and taking the limit as $\epsilon \to 0^+$, we obtain inequality (11). If $c_{a,t+a} > 0$ we can consider $\epsilon < 0$. Dividing through by $\epsilon < 0$ and taking the limit as $\epsilon \to 0^-$, we obtain inequality (12). □

Main results

Lemma 2 *If $\boldsymbol{x}$ is an oss it must be of the form $\boldsymbol{x}(z)$ for some value of $z \in [0, 1]$.*

Proof We will prove that only the σ-age class can be harvested along an optimal stationary path. This implies that if the total forest area is z, the state must be $\boldsymbol{x}(z)$.[25] We will assume that along a stationary path every harvested fraction of land is immediately replanted with trees to be harvested at the exact same age as that of their predecessors.[26]

Assume that $\boldsymbol{x}$ is an oss, i.e., the optimal path starting from it is constant $\boldsymbol{x}_t = \boldsymbol{x}$. Of course, along this path, harvest is constant, let us denote the total consumption by C. To reach a contradiction, let us assume that there is some age $s \neq \sigma$ such that $c_s > 0$. We consider an ϵ area of land that at $t = 0$ is planted with trees that will be harvested at age s and modify the stationary path by harvesting this area exactly when the age of the trees is σ. This means that every σ years, the harvest of the σ age class is increased by ϵ and that every s years, the harvest of the s age class is reduced by ϵ.

The marginal benefit of the perturbed path with respect to the constant one is

$$-\frac{b^s f_s}{1-b^s} u'(C) + \frac{b^\sigma f_\sigma}{1-b^\sigma} u'(C) > 0$$

that is strictly positive due to A_4. In consequence, it is not optimal to harvest from $s \neq \sigma$. ☐

Proposition 1 *The set of oss coincides with* $\{\boldsymbol{x}(z)/z \in [z_\delta, z_0]\}$, *i.e., the set of states with forest cover* $z \in [z_\delta, z_0]$ *where the forest area* z *is evenly divided among trees of age classes* 1 *through* σ. *In particular,*

1. *If* $\Delta(0) \leq 0$, *then* $I(\delta) = \{0\}$ *for all* $\delta \geq 0$, *i.e., regardless of the extent of irreversibility (cost of reforestation) the totally deforested state is the unique oss.*
2. *If* $\Delta(1) \geq \delta$, *then* $I(\delta) = \{1\}$, *i.e., there is a unique oss and it is characterized by full forest cover.*
3. *If* $\delta > 0$, $\Delta(0) > 0$ *and* $\Delta(1) < \delta$, *then* $z_\delta < z_0$ *and there is a continuum of oss; the set of forest covers that can be sustained in an oss is a closed interval* $[z_\delta, z_0]$ *with non-empty interior. In addition,* $I(\delta) = \{z \in [0, 1] : 0 \leq \Delta(z) \leq \delta\}$.

Proof The proof will be divided into two parts:

- First, we show: if $\boldsymbol{x}$ is an oss $\Rightarrow \boldsymbol{x} = \boldsymbol{x}(z)$ and $z \in [z_\delta, z_0]$
- Second, we show the counterpart: if $\boldsymbol{x} = \boldsymbol{x}(z)$ and $z \in [z_\delta, z_0] \Rightarrow \boldsymbol{x}$ is an oss
- **First part.** Thanks to Lemma 2 we already know that $\boldsymbol{x}$ is of the form $\boldsymbol{x}(z)$, so we only need to show that $z \in [z_\delta, z_0]$. We claim that

$$\text{If } z \in I \text{ with } z < 1 \text{ then } \Delta(z) \leq \delta \tag{13}$$

$$\text{If } z \in I \text{ with } z > 0 \text{ then } \Delta(z) \geq 0. \tag{14}$$

[25] In our model, no optimal stationary path has trees dying naturally. This is very intuitive, harvesting a tree before it dies and (instead of planting it) dedicating the land to the alternative use during what would have been its lifetime, provides a higher benefit.

[26] As our model is not spatial, we cannot actually impose it but we find that our proof is clearer under this additional "condition."

Postponing briefly the proof of (13) and (14), we use them to show that $z \in [z_\delta, z_0]$. Observe that the definition of z_0 and z_δ can be stated as $z_q = \max\{0, \min\{1, \Delta^{-1}(q)\}\}$ for $q = 0, \delta$.

1. if $z = 1$ we know thanks to (14) that $\Delta(1) \geq 0$. Since $\Delta(z)$ is a decreasing function, this condition implies $1 \leq \Delta^{-1}(0)$. By definition $z_0 = 1$ and evidently $z \in [z_\delta, z_0]$.
2. if $z = 0$, (13) implies that $\Delta(0) \leq \delta$. We then have that $0 \geq \Delta^{-1}(\delta)$. By definition $z_\delta = 0$ and evidently $z \in [z_\delta, z_0]$.
3. We deal now with the more general case, where $0 < z < 1$. Using both (13) and (14), we get $0 \leq \Delta(z) \leq \delta$ and the monotonicity of Δ yields $\Delta^{-1}(0) \geq z \geq \Delta^{-1}(\delta)$. Combining these inequalities, we get

$$z \leq \min\{1, \Delta^{-1}(0)\} \Rightarrow z \leq z_0$$
$$z \geq \min\{0, \Delta^{-1}(\delta)\} \Rightarrow z \geq z_\delta$$

This completes the proof of the first part. We present now the rather technical proof of (13) and (14).

We define the unit vector $\boldsymbol{e}_j \in \mathbb{R}^n$ such that $e_j = 1$ and $e_k = 0$ for all $k \neq j$ and $t(\sigma)$ stands for the remainder of the integer division of t by σ. We define a program alternative to the constant one starting at $\boldsymbol{x}(z)$, $\{\widehat{\boldsymbol{x}}_s\}_{s=0}^{\infty}$ such that

$$\widehat{\boldsymbol{x}}_0 = \boldsymbol{x}_0 \qquad \text{and} \qquad \widehat{\boldsymbol{x}}_t = \boldsymbol{x}_t + \epsilon \boldsymbol{e}_{t(\sigma)}\ \forall t \geq 1.$$

In words, whenever $\epsilon > 0$ ($\epsilon < 0$), the alternative program is one where the area under alternative use is reduced (increased) by ϵ at stage $t = 0$ and replanted to yield more (less) young forest next period. There is no further reallocation of land; and at every period, only the σ-age class is clearcut leaving the other classes untouched. This policy generates a σ-periodic path from $t = 1$ onward.

Of course, the area under alternative use can only be reduced if it is strictly positive, i.e., $1 - z > 0 \Rightarrow z < 1$. On the other hand, to increase the land under alternative use, we decrease the area of one of the age classes; hence, we need $z > 0$.

As the constant path is optimal, the modification cannot be gainful. Proceeding analogously to the proof of Lemma 6 we obtain,

$$\frac{b^\sigma}{1-b^\sigma}\left[u\left(\frac{f_\sigma}{\sigma}z\right) - u\left(f_\sigma\left(\frac{z}{\sigma}+\epsilon\right)\right)\right] + \frac{b}{1-b}[w(y) - w(y-\epsilon)] + v(\epsilon) - v(0) \geq 0.$$

If $z < 1$ we can consider $\epsilon > 0$. Dividing through by ϵ and taking the limit as $\epsilon \to 0^+$, we obtain (13):

$$-\frac{b^\sigma f_\sigma}{1-b^\sigma}u'\left(\frac{f_\sigma}{\sigma}z\right) + \frac{b}{1-b}w'(y) + \delta \geq 0 \Rightarrow \delta \geq \Delta(z)$$

If $z > 0$ we can consider $\epsilon < 0$. This time, dividing through by ϵ involves changing the sense of the inequality. Taking the limit as $\epsilon \to 0^-$, we obtain (14):

$$-\frac{b^\sigma f_\sigma}{1-b^\sigma} u'\Big(\frac{f_\sigma}{\sigma} z\Big) + \frac{b}{1-b} w'(y) \le 0 \Rightarrow 0 \le \Delta(z).$$

– **Second part.** Considering the definition of z_0 and z_δ, we have the following equivalences

(a) $[z_\delta, z_0] = \{0\}$ if, and only if, $\Delta(0) \le 0$.
(b) $[z_\delta, z_0] = \{1\}$ if, and only if, $\Delta(1) \ge \delta$.
(c) $[z_\delta, z_0] = \{z \in [0, 1] : 0 \le \Delta(z) \le \delta\}$ if, and only if, $\delta > 0$, $\Delta(0) > 0$ and $\Delta(1) < \delta$.

We will use these equivalences to prove the second part of the Proposition. In particular, we claim that

$$(a)\quad \text{If } \Delta(0) \le 0 \text{ then } \boldsymbol{x}(0) \text{ is a oss} \tag{15}$$

$$(b)\quad \text{If } \Delta(1) \ge \delta \text{ then } \boldsymbol{x}(1) \text{ is a oss} \tag{16}$$

$$(c)\quad \text{If } \Delta(z) \in [0, \delta] \text{ then } \boldsymbol{x}(z) \text{ is a oss (including the cases } z = 0 \text{ and } z = 1) \tag{17}$$

These statements can be intuitively interpreted by recalling that $\Delta(z)$ represents the marginal benefit of dedicating more land to forestry when the current state is $\boldsymbol{x}(z)$. Indeed, $\Delta(0) \le 0$ implies that, even when all the land is allocated to the alternative use, there is no incentive to dedicate land to forestry because the marginal benefit of such decision is negative. In other words, it means that any state $\boldsymbol{x}(z)$ with $z > 0$ cannot be sustainable because the marginal benefit of transferring land to the alternative use would be positive. In consequence, $\boldsymbol{x}(0)$ is the only eventual oss. The inequality $\Delta(1) \ge \delta$ implies that the marginal benefit of transferring land from alternative use to forestry is higher than the intrinsic marginal reforestation cost (δ). This is so, even when all the land is already allocated to forestry, and the marginal benefit of this transference is at its minimum, implying that no state $\boldsymbol{x}(z)$ with less than full forest cover can be a oss. Finally, in case (c) where $\Delta(z) \in [0, \delta]$ we see that the marginal benefit of transferring land to forestry is not enough to cover the intrinsic marginal reforestation costs and that the marginal benefit of relocating land to the alternative use is, at best, non-positive. Hence, $\boldsymbol{x}(z)$ is a promising candidate to oss.
We will use this intuitive interpretation to define the dual variables of two auxiliary variables accounting for reforestation (r_t) and deforestation (d_t). The new variables r_t and d_t are both nonnegative and satisfy $z_{t+1} = z_t + r_t - d_t$ for all t. By introducing them, we avoid dealing with the non-differentiable function ν. Indeed, We can substitute $\nu(z_{t+1} - z_t)$ by the term $\tilde{\nu}(r_t)$ where $\tilde{\nu}(r)$ is a smooth and concave function that coincides with $\nu(r)$ for nonnegative values of r. We have then $\tilde{\nu}'(0) = \delta$.

Eliminating the variables $c_{a,t}$ and c_t from the optimization problem (6), we get

$$\begin{cases} \text{maximize} & \sum_{t=0}^{\infty} b^t [u(\sum_{a=1}^{n-1} f_a(x_{a,t} - x_{a+1,t+1}) + f_n x_{n,t}) + w(y_t) - \tilde{v}(r_t)] \\ \text{subject to} & \sum_{a=1}^{n} x_{n,t} + y_t \le 1 \quad \forall t \\ & x_{a,t} - x_{a+1,t+1} \ge 0 \quad a = 1, \ldots, n-1, \ \forall t \\ & y_{t+1} = y_t - r_t + d_t \quad \forall t \\ & x_{n,t}, y_t, r_t, d_t \ge 0 \quad \forall t \end{cases} \tag{18}$$

And the associated Lagrangian is:

$$\begin{aligned} \mathcal{L} = \sum_{t=0}^{\infty} b^t \Bigg[u\left(\sum_{a=1}^{n-1} f_a(x_{a,t} - x_{a+1,t+1}) + f_n x_{n,t} \right) + w(y_t) \\ - \tilde{v}(r_t) \Bigg] + \theta_t (1 - \sum_a x_{a,t} - y_t) \\ + \sum_{a=1}^{n-1} \rho_{a,t}(x_{a,t} - x_{a+1,t+1}) + \mu_t(y_{t+1} - y_t + r_t - d_t) + \lambda_t^x x_{n,t} \\ + \lambda_t^y y_t + \lambda_t^r r_t + \lambda_t^d d_t. \end{aligned}$$

The partial derivatives of $\mathcal{L}$ evaluated along the constant path are:

$$\begin{aligned} \mathcal{L}_{x_{1,t}} &= b^t f_1 u'\left(\frac{f_\sigma}{\sigma} \hat{z}\right) - \theta_t + \rho_{1,t} \\ \mathcal{L}_{x_{a,t}} &= (b^t f_a - b^{t-1} f_{a-1}) u'\left(\frac{f_\sigma}{\sigma} \hat{z}\right) - \theta_t + \rho_{a,t} - \rho_{a-1,t-1} \qquad \forall a = 2, \ldots, n-1 \\ \mathcal{L}_{x_{n,t}} &= (b^t f_n - b^{t-1} f_{n-1}) u'\left(\frac{f_\sigma}{\sigma} \hat{z}\right) - \theta_t - \rho_{n-1,t-1} + \lambda_t^x \\ \mathcal{L}_{y_t} &= b^t w'(\hat{y}) - \theta_t + \mu_{t-1} - \mu_t + \lambda_t^y \\ \mathcal{L}_{r_t} &= -b^t \delta + \mu_t + \lambda_t^r \\ \mathcal{L}_{d_t} &= -\mu_t + \lambda_t^d \end{aligned}$$

To prove that $\boldsymbol{x}(z)$ is a oss, we need to show that the constant path is a solution of the optimization problem (18) when starting from it.
We first deal with (17). To show that the stationary solution is optimal, we propose the following set of ℓ_1 multipliers where τ_a stands for $\frac{b^a}{1-b^a}$:

$$\begin{cases} \theta_t = b^t \frac{f_\sigma \tau_\sigma}{\tau_1} u'\left(\frac{f_\sigma}{\sigma}\hat{z}\right) \\ \rho_{a,t} = \frac{b^t}{\tau_a}(\tau_\sigma f_\sigma - \tau_a f_a) u'\left(\frac{f_\sigma}{\sigma}\hat{z}\right) \\ \mu_t = b^t \Delta(\hat{z}) \\ \lambda_t^r = b^t [\delta - \Delta(\hat{z})] \\ \lambda_t^d = b^t \Delta(\hat{z}) \\ \lambda_t^x = b^t \frac{f_\sigma \tau_\sigma}{\tau_n} u'\left(\frac{f_\sigma}{\sigma}\hat{z}\right) \\ \lambda_t^y = 0 \end{cases}$$

Due to the definition of σ (see A_4), complementarity slackness and the non-negativity of all the multipliers are satisfied.[27] A straightforward computation shows that $\nabla \mathcal{L} = 0$ so the proposed path is a stationary point of $\mathcal{L}$. As the optimization problem (6) is strictly concave, optimality follows.
The proof of (16) is analogous. In this case, we propose the following set of ℓ_1 nonnegative multipliers,

$$\begin{cases} \theta_t = b^t \frac{f_\sigma \tau_\sigma}{\tau_1} u'(f_\sigma/\sigma) \\ \rho_{a,t} = \frac{b^t}{\tau_a}(\tau_\sigma f_\sigma - \tau_a f_a) u'(f_\sigma/\sigma) \\ \lambda_t^d = b^t \delta \\ \lambda_t^x = b^t \frac{f_\sigma \tau_\sigma}{\tau_n} u'(f_\sigma/\sigma) \\ \lambda_t^y = b^t \tau_1 \Delta(1) \\ \mu_t = \lambda_t^r = 0 \end{cases}$$

and the verification follows the same lines of the previous case.

Finally, we deal with (15). From the results in Piazza and Roy (2015), we know that if $\Delta(0) \leq 0$ then $\boldsymbol{x}(0)$ is an oss in the absence of reforestation costs. The stationary path remaining at the total deforestation state will not see its total discounted utility modified when deforestation costs are introduced. It is evident that no other path will yield higher total discounted utility with the introduction of reforestation costs. In consequence, the total deforestation state continues to be an oss with reforestation costs and (15) follows. □

Lemma 4 *Suppose that $\Delta(0) \leq \delta$. Let $\{x_t\}$ be an optimal path and let $\{z_t\}$ be the path of area under forest cover associated with it. Then, $z_t \geq z_{t+1}$ for all t, i.e., the area under forest cover is always non-increasing along an optimal program. If, further, $\Delta(0) > 0$, then*

$$z_t \geq \min\{\underline{z}, z_o\} \text{ for all } t.$$

[27] With the exception of μ_t that, being associated with an equality constraint, does not have any sign requirements.

Proof To prove that $\Delta(0) \leq \delta \Rightarrow z_t$ is non-increasing, we proceed by contradiction. Assume that there exists at least one optimal path, such that the forest cover increases at least once. Without loss of generality, we assume that the increase takes place at $t = 0$.

If $z_o < z_1$, then $x_{1,1} > 0$ and there is a such that $c_{a,a} > 0$. We apply then (12)

$$
\begin{aligned}
0 &\geq \sum_{j=1}^{a} b^j w'(y_j) + \underbrace{\nu'(z_1 - z_0)}_{>\delta} - b^a f_a u'(c_a) - b^a \nu'_+(z_{a+1} - z_a) \\
&> \frac{b(1-b^a)}{1-b} w'(1) - b^a f_a u'(0) + \delta - b^a \nu'_+(z_{a+1} - z_a) \\
&= (1-b^a)\left[\frac{b}{1-b} w'(1) - \frac{b^a f_a}{1-b^a} u'(0)\right] + \delta - b^a \nu'_+(z_{a+1} - z_a) \\
&\geq -(1-b^a)\Delta(0) + \delta - b^a \nu'_+(z_{a+1} - z_a)
\end{aligned}
$$

We have different scenarios depending on the change of the forest cover at $t = a$. For instance, if $z_a > z_{a+1}$ we have $\nu'(z_{a+1} - z_a) = 0$ and if $z_a = z_{a+1}$ we have $\nu'_+(z_{a+1} - z_a) = \delta$. In both cases, we conclude $0 > -(1-b^a)\Delta(0) + \delta - b^a \nu'_+(z_{a+1} - z_a) \geq -(1-b^a)\Delta(0) + (1-b^a)\delta$. This implies that $\Delta(0) > \delta$, contradicting $\Delta(0) \leq \delta$ and proving the first statement of the lemma.

The case where $z_a < z_{a+1}$ is the difficult one, because we may have $\nu'(z_{a+1} - z_a) > \delta$. Before going further, we recall the deduction of the Euler inequality (12). It was based on a perturbation where the area under alternative use is increased by ϵ at stage $t+1$ and less young forest is replanted. This modification of the young forest area is allowed to grow undisturbed until age a at which point the consumption of forest of age a is modified. To compensate for the perturbation, the area under alternative use can only be increased by decreasing harvest of age class a at stage $t = a$; hence, we need $c_{a,a} > 0$. But if $z_a < z_{a+1}$, then $x_{a+1,1} > 0$ and there exists a' such that $c_{a+a',a'} > 0$. We modify the perturbation above keeping the extra ϵ of land allocated to the alternative use for another a' stages. Observe that the term $\nu'_+(z_{a+1} - z_a)$ disappears because land is *not* returned to forest use.

We then have

$$
\begin{aligned}
0 &\geq \sum_{j=1}^{a+a'} b^j w'(y_j) - b^a f_a u'(c_a) - b^{a+a'} f_{a'} u'(c_{a+a'}) \\
&\quad + \nu'(z_1 - z_0) - b^{a+a'} \nu'_+(z_{a+a'+1} - z_{a+a'}) \\
&> (1-b^a)\left[\frac{b}{1-b} w'(1) - \frac{b^a f_a}{1-b^a} u'(0)\right] \\
&\quad + b^a(1-b^{a'})\left[\frac{b}{1-b} w'(1) - \frac{b^{a'} f_{a'}}{1-b^{a'}} u'(0)\right] + \delta - b^{a+a'} \nu'_+(z_{a+1} - z_a) \\
&> -\Delta(0)(1-b^{a+a'}) + \delta - b^{a+a'} \nu'_+(z_{a+a'+1} - z_{a+a'})
\end{aligned}
$$

If $z_{a+a'} \geq z_{a+a'+1}$, the proof is complete because $\nu'_+(z_{a+a'+1} - z_{a+a'}) \leq \delta$ and again $\Delta(0) > \delta$ is reached.

If not, we repeat the process. Let us introduce some notation for the last part of the proof. We denote $a_1 = a$ and $a_2 = a'$ and $s_j = \sum_{i \leq j} a_i$. If $z_{s_j} \leq z_{s_j+1}$ then $x_{s_j+1,1} > 0$ and there exists a_{j+1} such that $c_{s_j+a_{j+1},a_{j+1}} > 0$. We modify the perturbation above keeping the extra ϵ of land allocated to the alternative use for another a_{j+1} stages. We repeat the process until we reach some stage such that $z_{s_j} > z_{s_j+1}$.

If such stage is not reached, then the process is repeated infinitely to get

$$
\begin{aligned}
0 &\geq \lim_{J\to\infty} \sum_{j=1}^{J} b^{s_{j-1}}(1-b^{a_j}) \left[\frac{b}{1-b} w'(1) - \frac{b^{a_j} f_{a_j}}{1-b^{a_j}} u'(0) \right] \\
&\quad + \nu'(z_1 - z_0) - \lim_{J\to\infty} b^{s_J} \nu'_+(z_{s_J+1} - z_{s_J}) \\
&> - \lim_{J\to\infty} \sum_{j=1}^{J} b^{s_{j-1}}(1-b^{a_j})\Delta(0) + \delta - \lim_{J\to\infty} b^{s_J} \nu'_+(z_{s_J+1} - z_{s_J}) \\
&= -\Delta(0)\big(1 - \lim_{J\to\infty} b^{s_J}\big) + \delta - \lim_{J\to\infty} b^{s_J} \nu'_+(z_{s_J+1} - z_{s_J}) \\
&= -\Delta(0) + \delta
\end{aligned}
$$

a contradiction. The proof follows.

We proceed now to prove that $0 < \Delta(0) \leq \delta \Rightarrow z_t \geq \min\{\underline{z}, z_o\}$ for all t. Given the monotonicity of z_t, there are only two possibilities: either (i) $z_t = z_o$ for all t or (ii) there exists t such that $z_{t-1} > z_t$. The proposition follows trivially in (i).

In (ii), let T be any stage where the forest cover decreases, i.e., $z_{T-1} > z_T$. We know that $z_t \leq z_T < z_{T-1} \leq 1$ for all $t > T$; hence, $z_{T-1+j} < 1$ for all $j = 1, \ldots, \sigma$ and then $\min_{j=1,\ldots,\sigma}\{y_{T-1+j}\} > 0$. We apply (11) to get

$$
\begin{aligned}
0 &\leq \sum_{j=1}^{\sigma} b^j w'(y_{T-1+j}) + \nu'_+(z_T - z_{T-1}) - b^\sigma u'(c_{T-1+\sigma}) - b^\sigma \nu'_-(z_{T+\sigma} - z_{T-1+\sigma}) \\
&= \sum_{j=1}^{\sigma} b^j w'(y_{T-1+j}) - b^\sigma u'(c_{T-1+\sigma}) \\
&\leq \sum_{j=1}^{\sigma} b^j w'(y_{T-1+j}) - b^\sigma u'(f_m(1 - y_{T-1+\sigma})) \\
&\leq \frac{b(1-b^\sigma)}{1-b} w'(1 - z_T) - b^\sigma u'(f_m(z_T))
\end{aligned}
$$

Implying that $g(z_T) \leq 0$ and, in consequence, $z_T \geq \underline{z}$. □

Proposition 4 *There is global weak conservation if, and only if,* $\Delta(0) > 0$.

Proof It is sufficient to show that

$$\Delta(0) \leq 0 \Leftrightarrow \text{Global weak conservation does not hold.}$$

If $\Delta(0) \leq 0$, the results in Piazza and Roy (2015) assure that every optimal path is characterized by immediate deforestation *in the absence of reforestation costs*. Evidently, when reforestation costs are introduced, any path characterized by immediate deforestation will not change its total benefit, while the total benefit of the other paths will either remain constant or decrease. Hence, if $\Delta(0) \leq 0$ the optimal paths are the same with and without reforestation costs. And they are all characterized by immediate and eventual total deforestation.

If there is no global weak conservation, there exists at least one optimal path, $\{x_t\}$, starting from a positive forest cover state characterized by eventual total deforestation.

Along $\{x_t\}$, we know that $z_t \to 0$. This convergence can be finite or asymptotic, i.e., either there exists T such that $z_T > 0$ and $z_t = 0$ for all $t > T$ or there exists a subsequence $\{t_k\}_{k\in I\!N}$ such that $z_{t_k-1} > z_{t_k}$.

In the first case, we apply (11) for $t = T$ and $a = \sigma$ to get,

$$0 \leq \frac{b(1-b^\sigma)}{1-b} w'(1) - b^\sigma f_\sigma u'(0) = -(1-b^\sigma)\Delta(0) \Rightarrow \Delta(0) \leq 0.$$

In the second case, we evaluate (11) along the subsequence $\boldsymbol{x}_{t_k}$ and $a = \sigma$,

$$\begin{aligned} 0 &\leq \sum_{j=1}^{\sigma} b^j w'(y_{t_k+j}) - b^\sigma f_\sigma u'(c_{t_k+\sigma}) - b^\sigma v'_-(z_{t_k+\sigma+1} - z_{t_k+\sigma}) \\ &\leq \sum_{j=1}^{\sigma} b^j w'(y_{t_k+j}) - b^\sigma f_\sigma u'(c_{t_k+\sigma}) \end{aligned}$$

and letting $k \to \infty$ we get $0 \leq -(1-b^\sigma)\Delta(0) \Rightarrow \Delta(0) \leq 0$. □

Proposition 5 *The following are equivalent*

1. *There is global weak conservation*
2. $\Delta(0) > 0$
3. *There exists at least one oss with strictly positive forest cover*
4. *There exists at least one optimal path where eventual total deforestation does not occur*
5. *There exists at least one optimal path where immediate total deforestation does not occur*

Proof The equivalence 1.⇔ 2. is exactly Proposition 4 and Corollary 4 delivers 2.⇔ 3. To complete the proof, we show that 3.⇒ 4.⇒ 5.⇒ 2., which are all fairly straightforward.

Given a positive forest cover oss, $\boldsymbol{x}(\hat{x})$, the constant path starting from it is not characterized by eventual total deforestation, proving 3.⇒ 4.

4.⇒ 5. follows trivially. Indeed, any path not characterized by eventual total deforestation is not characterized by immediate total deforestation, either.

It is only left to show that 5.⇒ 2., and we proceed by contradiction. If $\Delta(0) \leq 0$, we know thanks to Piazza and Roy (2015), that every optimal program is characterized by immediate total deforestation in the absence of reforestation costs. Following the same reasoning of the proof of Proposition 4, we conclude that all the optimal paths are characterized by immediate deforestation in the presence of such costs, contradicting 5. □

Proposition 6 *If $\{x_t\}$ is an optimal program and $\{z_t\}$ is its associated path of area under forest cover, then for every $t \geq 0$,*

$$\max_{j=1,\ldots,\sigma} \{z_{t+j}\} \geq \min\{\underline{z}, z_o\}$$

Proof We first note that if $\max_{j=1,\ldots,a}\{z_{t+j}\} = 1$, then (7) holds trivially. From now on, we assume that the maximum is less than one which allows us to use (11) for $a = \sigma$.

We proceed by induction on t. First consider $t = 0$. If $z_1 \geq z_o$, the inequality follows trivially. If not, we have $v'_+(z_1 - z_o) = 0$ and (11) implies,

$$\begin{aligned} 0 &\leq \sum_{j=1}^{\sigma} b^j w'(1 - z_j) - b^\sigma f_\sigma u'(c_\sigma) - b^\sigma v'_-(z_{\sigma+1} - z_\sigma) \\ &< \frac{b(1 - b^\sigma)}{1 - b} w'(1 - \max_{j=1,\ldots,\sigma} \{z_j\}) - b^\sigma f_\sigma u'(f_m \max_{j=1,\ldots,\sigma} \{z_j\}) \end{aligned}$$

Considering the definition of $\underline{z}$, we get that $\max_{j=1,\ldots,\sigma}\{z_j\} > \underline{z}$, and hence, (7) holds for $t = 0$.

We now assume that (7) holds for $t = k - 1$ and prove that it is true for $t = k$.

If $z_{k+1} < z_k$, we apply (11) at $t = k$ and follow the same steps of the case $t = 0$ to conclude.

If $z_{k+1} \geq z_k$, then

$$\max_{j=1,\ldots,\sigma} \{z_{k-1+j}\} = \max_{j=2,\ldots,\sigma} \{z_{k-1+j}\} \leq \max_{j=2,\ldots,\sigma+1} \{z_{k-1+j}\} = \max_{j=1,\ldots,\sigma} \{z_{k+j}\}.$$

In consequence, $\min\{\underline{z}, z_o\} \leq \max_{j=1,\ldots,\sigma}\{z_{k-1+j}\} \leq \max_{j=1,\ldots,\sigma}\{z_{k+j}\}$ where the first inequality is (7) for $t = k - 1$. The proof follows by induction on t. □

Proposition 7 *If $\{x_t\}$ is an optimal program and $\{z_t\}$ is its associated path of area under forest cover, then*

$$\liminf_t z_t < \bar{z}(0).$$

Proof To obtain a contradiction, suppose that there is T such that the optimal program satisfies $z_t \geq \bar{z}(0)$ for all $t \geq T$. Take $t \geq T$ with $x_{t+1,1} > 0$.[28] As a consequence, there exists a such that $c_{t+a,a} > 0$, using (12) we get

$$
\begin{aligned}
0 &\geq \sum_{j=1}^{a} b^j w'(1 - z_{t+j}) - b^a f_a u'(c_{t+a}) + \underbrace{\nu'_-(z_{t+1} - z_t)}_{\geq 0} - b^a \nu'_+(z_{t+a+1} - z_{t+a}) \\
&> \frac{b(1-b^a)}{1-b} w'(1 - \bar{z}(0)) - b^a f_a u'(0) - b^a \nu'_+(z_{t+a+1} - z_{t+a}) \\
&\geq (1 - b^a)\left[\frac{b}{1-b} w'(1 - \bar{z}(0)) - \frac{b^a f_a}{1 - b^a} u'(0)\right] - b^a \nu'_+(z_{t+a+1} - z_{t+a}) \\
&\geq (1 - b^a)\left[\frac{b}{1-b} w'(1 - \bar{z}(0)) - \frac{b^\sigma f_\sigma}{1 - b^\sigma} u'(0)\right] - b^a \nu'_+(z_{t+a+1} - z_{t+a})
\end{aligned}
$$

The term between brackets is zero due to (8). If $z_{t+a} > z_{t+a+1}$ then $\nu'_+(z_{t+a+1} - z_{t+a}) = 0$ implying that the last term is zero and a contradiction is reached.

The case where $z_{t+a} \leq z_{t+a+1}$ is the difficult one. The proof is analogous to that of Lemma 4. Indeed, if $z_{t+a} \leq z_{t+a+1}$, then $x_{t+a+1,1} > 0$ and there exists a' such that $c_{t+a+a',a'} > 0$. We consider then a perturbation where the area under alternative use is increased by ϵ at stage $t+1$ and the fraction of land is only returned to the forestry use after $a + a'$ time steps. This causes a reduction in ϵ of the consumption of age class a at stage $t + a$ and of age class a' at stage $t + a + a'$.

We get then

$$
\begin{aligned}
0 &\geq \sum_{j=1}^{a+a'} b^j w'(1 - z_{t+j}) - b^a f_a u'(c_{t+a}) - b^{a+a'} f_{a'} u'(c_{t+a+a'}) \\
&\quad + \nu'_-(z_{t+1} - z_t) - b^{a+a'} \nu'_+(z_{t+a+a'+1} - z_{t+a+a'}) \\
&> (1 - b^a)\left[\frac{b}{1-b} w'(1 - \bar{z}(0)) - \frac{b^\sigma f_\sigma}{1 - b^\sigma} u'(0)\right] \\
&\quad + b^a(1 - b^{a'})\left[\frac{b}{1-b} w'(1 - \bar{z}(0)) - \frac{b^\sigma f_\sigma}{1 - b^\sigma} u'(0)\right] - b^{a+a'} \nu'_+(z_{t+a+1} - z_{t+a}) \\
&= -b^{a+a'} \nu'_+(z_{t+a+a'+1} - z_{t+a+a'})
\end{aligned}
$$

If $z_{t+a+a'} > z_{t+a+a'+1}$, the proof is complete because $\nu'_+(z_{t+a+a'+1} - z_{t+a+a'}) = 0$ and again a contradiction is reached.

If not, we repeat the process until we reach some stage such that the forest cover is decreased when the perturbed fraction of land is returned to the forestry use, i.e., until we reach some stage such that $z_{t+s_j} > z_{t+s_j+1}$.[29]

[28] If there is no t such that $x_{t+1,1} > 0$ then $z_t \to 0$ contradicting our assumption.

[29] At this point, we have adopted the notation introduced in the proof of Lemma 4.

If such stage is not reached, then the process is repeated infinitely to get

$$
\begin{aligned}
0 \geq & \lim_{J\to\infty} \sum_{j=1}^{J} b^{s_j-1}[\sum_{i=1}^{a_j} b^i w'(1-z_{t+s_j+i}) - b^{a_j} f_{a_j} u'(c_{s_j})] + \nu'_-(z_{t+1}-z_t) \\
& - \lim_{J\to\infty} b^{s_J} \nu'_+(z_{t+s_J+1} - z_{t+s_J}) \\
> & \lim_{J\to\infty} \sum_{j=1}^{J} b^{s_j-1}(1-b^{a_j}) \left[\frac{b}{1-b} w'(\bar{z}(0)) \right. \\
& \left. - \frac{b^\sigma f_\sigma}{1-b^\sigma} u'(0) \right] - \lim_{J\to\infty} b^{s_J+1} \nu'_+(z_{t+s_J+1} - z_{t+s_J}) \\
= & - \lim_{J\to\infty} b^{s_J} \nu'_+(z_{t+s_{J+1}+1} - z_{t+s_{J+1}}) = 0.
\end{aligned}
$$

a contradiction. The proof follows. □

Proposition 8 *If $\{x_t\}$ is an optimal program, $\{z_t\}$ is its associated path of area under forest cover and there is T such that $z_T < z_{T+1}$ (forest cover increases in period T), then*

$$
\min_{j=1,\dots,n} \{z_{t+j}\} \leq \bar{z}(\delta) \text{ for all } t \geq T.
$$

Proof If $z_T < z_{T+1}$, we know that $x_{T+1,1} > 0$. The trees planted at T must be harvested between $T+1$ and $T+n$. Hence, there exists $a \in [1,n]$ such that $c_{T+a,a} > 0$ and using (12)

$$
\begin{aligned}
0 \geq & \sum_{j=1}^{a} b^j w'(1-z_{T+j}) + \underbrace{\nu'_-(z_{T+1}-z_T)}_{\geq\delta} - b^a f_a u'(c_{T+a}) - b^a \nu'_+(z_{T+a+1} - z_{T+a}) \\
> & \frac{b(1-b^a)}{1-b} w'\big(1 - \min_{j=1,\dots,a}\{z_{T+j}\}\big) - b^a f_a u'(0) + \delta - b^a \nu'_+(z_{T+a+1} - z_{T+a})
\end{aligned}
$$

We have different scenarios depending on the change in the forest cover at $t = T + a$. For instance, if $z_{T+a} > z_{T+a+1}$ we have $\nu'(z_{T+a+1} - z_{T+a}) = 0$ and if $z_{T+a} = z_{T+a+1}$ we have $\nu'_+(z_{T+a+1} - z_{T+a}) = \delta$. In both cases, we have

$$
\begin{aligned}
0 \geq & \sum_{j=1}^{a} b^j w'(1-z_{T+j}) + \underbrace{\nu'_-(z_{T+1}-z_T)}_{\geq\delta} - b^a f_a u'(c_{T+a}) - b^a \nu'_+(z_{T+a+1} - z_{T+a}) \\
> & (1-b^a) \left[\frac{b}{1-b} w'(1 - \min_{j=1,\dots,a}\{1 - z_{T+j}\}) - \frac{b^a f_a}{1-b^a} u'(0) + \delta \right] \\
= & (1-b^a) \left[\frac{b}{1-b} w'(1 - \min_{j=1,\dots,n}\{1 - z_{T+j}\}) - \frac{b^\sigma f_\sigma}{1-b^\sigma} u'(0) + \delta \right]
\end{aligned}
$$

Considering the definition of $\bar{z}(\delta)$, we get $\min_{j=1,\ldots,n}\{z_{T+j}\} < \bar{z}(\delta)$, and hence, (9) holds for $t = T$. The case where $z_{T+a} < z_{T+a+1}$ is more difficult, but the proof follows the same lines as those of Lemma 4 and Proposition 7, details are left to the reader.

We now assume that (9) holds for $t = k - 1$ and prove that it is true for $t = k$.
If $z_k \leq z_{k+1}$, we apply (12) at $t = k$ and follow the same steps of the case $t = T$.
If $z_k > z_{k+1}$, then

$$\min_{j=1,\ldots,n}\{z_{k+j}\} = \min_{j=0,\ldots,n}\{z_{k+j}\} \leq \min_{j=0,\ldots,n-1}\{z_{k+j}\} = \min_{j=1,\ldots,n}\{z_{k-1+j}\} \leq \bar{z}(\delta)$$

where the last inequality is (9) for $t = k - 1$. The proof follows by induction on t. □

Lemma 5 *The solution to* (10), $\{z_t\}$, *satisfies the following:*

1. *If the initial forest cover z_o is larger or equal to z^s, the static optimal forest size, then the optimal path is $z_t = z^s$ for all $t \geq 1$.*
2. *If the initial forest cover z_o is below z^s, the optimal path is non-decreasing, bounded above by z^s and converges to a value in the interval $[z_o, z^s]$*

Proof 1. The policy $z_t = z^s$ for all $t \geq 1$ yields the maximum benefit when $\nu(\cdot) = 0$, and the reforestation costs do not affect its total benefit.

2. We prove first that $z_t \leq z^s$ for all t by contradiction. Assume that it is not true that $z_t \leq z^s$ for all t and take T to be lowest value of t such that $z_T > z^s$. From the paragraph above, we know that $z_t = z^s$ for all $t > T$.[30] We propose the following alternative path

$$\hat{z}_t = \begin{cases} z^s & \text{if } t = T \\ z_t & \text{if } t \neq T, \end{cases}$$

i.e., the forest cover at stage T is smaller along the alternative path than in the original one. The two paths coincide in every other time period.
The modification only affects the benefit at two stages: $t = T - 1$ and $t = T$, and we claim that it yields a strictly larger benefit. Indeed, at $t = T - 1$ we have

$$\begin{aligned} &u(f_1 z_{T-1}) + w(1 - z_{T-1}) - \nu(z_T - z_{T-1}) \\ &\quad < u(f_1 z_{T-1}) + w(1 - z_{T-1}) - \nu(z^s - z_{T-1}) \end{aligned}$$

where we are using that $z_T > \hat{z}_T = z^s \geq z_{T-1}$ which is equivalent to $z_T - z_{T-1} > z^s - z_{T-1} \geq 0$ and implies $\nu(z_T - z_{T-1}) > \nu(z^s - z_{T-1})$. And, at $t = T$ we have

$$u(f_1 z_T) + w(1 - z_T) - \nu(z^s - z_T) < u(f_1 z^s) + w(1 - z^s) - \nu(z^s - z^s)$$

where we are using that $z_{T+1} = \hat{z}_{T+1} = z^s$. Indeed, this last fact implies that $\nu(z^s - z_T) = \nu(z^s - z^s) = 0$, and, on the other hand, we have z^s is the unique

[30] We observe that this readily implies that the forest cover could exceed z^s at most at *one* stage, although this is not used in the proof.

maximum of $u(f_1 z) + w(1-z)$ which gives $u(f_1 z_T) + w(1-z_T) < u(f_1 z^s) + w(1-z^s)$. We have shown that the alternative path provides a strictly larger total benefit, reaching a contradiction.
We now show that the sequence is non-decreasing, again by contradiction. Assume that there is T such that $z_T < z_{T-1}$. We propose the following alternative path

$$\hat{z}_t = \begin{cases} z_{T-1} & \text{if } t = T \\ z_t & \text{if } t \neq T, \end{cases}$$

i.e., forest cover in stage T is larger along the alternative path than in the original one and the two paths coincide in every other time stage. The total benefit at stage $t = T-1$ is not modified because no reforestation costs are incurred along the two paths. However, at stage $t = T$ the benefit along $\{\hat{z}_t\}$ is greater. Indeed, given that $u(f_1 z) + w(1-z)$ is concave and maximized at z^s and the fact that $z_T < z_{T-1} \leq z^s$ we know that $u(f_1 z_T) + w(1-z_T) < u(f_1 z_{T-1}) + w(1-z_{T-1})$. Regarding the reforestation costs, we have that $z_{T-1} > z_T$ implies $\nu(z_{T+1} - z_T) \geq \nu(z_{T+1} - z_{T-1})$.[31] Thus,

$$\begin{aligned} &u(f_1 z_T) + w(1-z_T) - \nu(z_{T+1} - z_T) \\ &\quad < u(f_1 z_{T-1}) + w(1-z_{T-1}) - \nu(z_{T+1} - z_{T-1}). \end{aligned}$$

that is, $\{\hat{z}_t\}$ yields a strictly larger benefit in stage T. The rest of the stage utilities remain unchanged. This yields a contradiction.

□

Proposition 9 *Assume $n = 1$, then the following are equivalent*

1. *there is global strong conservation*
2. *there is global weak conservation*
3. $\Delta(0) > 0$.

Proof The proof of this proposition is as follows. By definition $1 \Rightarrow 2$. We prove next that $2 \Rightarrow 3$ and $3 \Rightarrow 1$.

The function $\Delta(z)$ is simply $\Delta(z) = \frac{b}{1-b}[f_1 u'(f_1 z) - w'(1-z)]$. Observe that $z^s = 0$ if and only if $\Delta(0) \leq 0$. We have then that total deforestation is optimal from every initial state if $\Delta(0) \leq 0$. Or, equivalently, that if there is a path such that total deforestation is not optimal, then $\Delta(0) > 0$. This is equivalent to $2 \Rightarrow 3$.

If $\Delta(0) > 0$, then $z^s > 0$. Lemma 5 implies that the forest cover remains always above $\min\{z_o, z^s\}$, which in turn implies that there is global strong conservation. □

References

Albers, H.J.: Modeling ecological constraints on tropical forest management: spatial interdependence, irreversibility, and uncertainty. J. Environ. Econ. Manag. **30**(1), 73–94 (1996)

[31] In fact, we can be even more precise: $\nu(z_{T+1} - z_T) > \nu(z_{T+1} - z_{T-1}) > 0$ if $z_{T+1} > z_{T-1}$, $\nu(z_{T+1} - z_T) > 0 = \nu(z_{T+1} - z_{T-1})$ if $z_T < z_{T+1} \leq z_{T-1}$ and $\nu(z_{T+1} - z_T) = \nu(z_{T+1} - z_{T-1}) = 0$ if $z_{T+1} \leq z_T$.

Albers, H.J., Fisher, A.C., Hanemann, W.M.: Valuation and management of tropical forest. Environ. Resour. Econ. **8**(1), 39–61 (1996)
Amacher, G.S., Ollikainen, M., Koskela, E.A.: Economics of Forest Resources. MIT Press, Cambridge (2009)
Angelsen, A., Kaimowitz, D.: Rethinking the causes of deforestation: lessons from economic models. World Bank Res. Obs. **14**, 73–98 (1999)
Arrow, K.J., Fisher, A.C.: Environmental preservation, uncertainty, and irreversibility. Q. J. Econ. **88**(2), 312–319 (1974)
Arrow, K.J., Kurz, M.: Optimal growth with irreversible investment in a Ramsey model. Econometrica **38**(2), 331–344 (1970)
Brock, W.A.: On existence of weakly maximal programs in a multi-sector economy. Rev. Econ. Stud. **37**(2), 275–280 (1970)
Cass, D.: Optimum growth in an aggregative model of capital accumulation. Rev. Econ. Stud. **32**(3), 233–240 (1965)
Clark, C.W.: Mathematical Bioeconomics: The Mathematics of Conservation, 3rd edn. Wiley, London (2010)
Dow Jr., J.P., Olson, L.J.: Irreversibility and the behavior of aggregate stochastic growth models. J. Econ. Dyn. Control **16**, 207–223 (1992)
Dupouey, J.-L., Dambrine, E., Laffite, J.D., Moares, C.: Irreversible impact of past land use on forest soils and biodiversity. Ecology **83**(11), 2978–2984 (2002)
Global Forest Resources Assessment. ISBN 978-92-5-109283-5. Food and Agriculture Organization of the United Nations (2015)
Fisher, A.C., Hanemann, W.M.: Option value and the extinction of species. In: Smith, V.K. (ed.) Advances in Applied Micro-Economics: Risk, Uncertainty and the Valuation of Benefits and Costs 4 169–190. JAI Press, Inc., Greenwich (1986)
Henry, C.: Option values in the economics of irreplaceable assets. Rev. Econ. Stud. **41**, 89–104 (1974)
Khan, M.A., Piazza, A.: On the Mitra–Wan forestry model: a unified analysis. J. Econ. Theory **147**(1), 230–260 (2012)
Majumdar, M., Nermuth, M.: Dynamic optimization in non-convex models with irreversible investment: monotonicity and turnpike results. J. Econ. **42**, 339–362 (1982)
Mäler, K.-G., Fisher, A.: Environment, uncertainty, and option values. Handb. Environ. Econ. **2**, 571–620 (2005)
Mitra, T., Wan Jr., H.Y.: Some theoretical results on the economics of forestry. Rev. Econ. Stud. **52**, 263–282 (1985)
Mitra, T., Wan Jr., H.Y.: On the Faustmann solution to the forest management problem. J. Econ. Theory **40**, 229–249 (1986)
Olson, L.J.: Stochastic growth with irreversible investment. J. Econ. Theory **47**, 101–129 (1989)
Olson, L.J.: Dynamic economic models with uncertainty and irreversibility: methods and applications. Nonlinear World **3**, 757–770 (1996)
Piazza, A., Roy, S.: Deforestation and optimal management. J. Econ. Dyn. Control **53**, 15–27 (2015)
Runyan, C.W., D'Odorico, P., Lawrence, D.: Physical and biological feedbacks of deforestation. Rev. Geophys. **50**(4), 1–32 (2012)
Sloan, S., Sayer, J.A.: Forest resources assessment of 2015 shows positive global trends but forest loss and degradation persist in poor tropical countries. For. Ecol. Manag. **352**, 134–145 (2015)
Sargent, T.J.: "Tobin's q" and the rate of investment in general equilibrium. In: Brunner, K., Meltzer, A.H. (eds.) On the State of Macroeconomics. North Holland, Amsterdam (1980)
Salo, S., Tahvonen, O.: On the economics of forest vintages. J. Econ. Dyn. Control **27**, 1411–1435 (2003)
Salo, S., Tahvonen, O.: Renewable resources with endogenous age classes and allocation of land. Am. J. Agric. Econ. **86**, 513–530 (2004)
Tahvonen, O.: Optimal harvesting of forest age classes: a survey of some recent results. Math. Popul. Stud. **11**(3–4), 205–232 (2004)

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