Optimality of Ramsey-Euler Policy in the Stochastic Growth Model*

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Abstract

For the canonical one sector stochastic optimal growth model, we outline a new set of conditions for a policy function that satisfies the Ramsey-Euler equation to be optimal. An interior Ramsey-Euler policy function is optimal if, and only if, it is continuous or alternatively, if, and only if, both consumption and investment are non-decreasing in output. In particular, we show that under these conditions, the stochastic paths generated by the policy must satisfy the transversality condition; the implication is that in *applying* our result, one does not need to verify the transversality condition when checking for optimality of a policy function.

Keywords: Stochastic growth, optimal economic growth, uncertainty, transversality condition, optimality, Euler equation.

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1 Introduction

The one sector optimal stochastic growth model (Brock and Mirman, 1972) is the canonical framework used by economists to examine problems of intertemporal resource allocation and more specifically, capital accumulation under uncertainty. It has been widely used as the basic model of macroeconomic growth under technology or productivity shocks and of optimal management of renewable natural resources affected by environmental uncertainty. Variations of the model have also been used to study business cycles.

In this model, a representative agent allocates the currently available output (of a single good) between investment and consumption where consumption generates immediate utility while investment generates next period's output according to a production function that is subject to exogenous production shocks. In the standard version of the model, the exogenous shocks are independent and identically distributed over time. The agent maximizes expected discounted sum of utility from consumption where the discount factor, the utility function and the production function are invariant over time. In such a stationary framework, the intertemporal economic trade-offs faced by the agent are reflected in the *optimal* consumption policy function. This function, which specifies the amount consumed as a function of current stock of output, has the property that when consumption over time is consistently chosen by following this policy, the path thereby generated is optimal among all paths feasible from the same initial stock. It is important to note that the consumption policy function specifies consumption as a function of current output, regardless of how and when that output level is reached. That is, it does not depend on the date at which the current output is observed, and it does not depend on the history of output levels reached before the current output is observed.

Conditions for optimality play a very important role in understanding the nature of this optimal policy function. In a large class of applications where economists work with specific functional forms for utility and production functions, sufficient conditions for optimality help determine whether an explicitly specified policy function is actually optimal. Even when one cannot derive explicit solutions to the dynamic optimization problem, sufficient conditions for optimality are useful in showing that a certain implicitly defined ("candidate") function is optimal. Necessary conditions for optimality are used to derive qualitative properties of optimal policy functions.

Optimality conditions for the dynamic optimization problem underlying the one sector stochastic growth model can also be useful in dynamic games of capital accumulation such as dynamic games of common property renewable resource extraction[1].

In a convex framework (strictly concave utility, concave production function), the existing literature has used duality theory to derive a set of conditions that are both necessary and sufficient for a policy function to be optimal and, in fact, to be the unique optimal policy function. In particular, an interior policy function (i.e., one where both consumption and investment are always strictly positive when the current stock of output is strictly positive) is optimal if, and only if, it satisfies the Euler condition (called the Ramsey-Euler equation in this literature) and a transversality condition (Mirman and Zilcha 1975, Zilcha 1976, 1978).[2,3]

The Ramsey-Euler equation is a simple first order condition that captures the trade-off between consumption in any two consecutive time periods, and takes the form of a functional equation. We refer to an interior consumption policy function satisfying this Ramsey-Euler equation as a *Ramsey-Euler policy* and this paper focuses on a systematic study of the optimality of such a policy.[4]

Using the characterization results mentioned above, a Ramsey-Euler policy can be shown to be an optimal policy, if it satisfies a transversality condition. The transversality condition essentially requires that the expected present value of capital stocks (valued by a shadow price equal to the discounted marginal utility of current consumption) converge to zero in the long run. It is an asymptotic condition on the entire stochastic process generated by the policy function.[5] Verifying the transver-

[1]See, for instance, Mitra and Sorger (2014).

[2]Key contributions emphasizing the importance of the transversality condition in models of intertemporal resource allocation include Malinvaud (1953), Cass (1965), Shell (1969), Peleg and Ryder (1972) and Weitzman (1973).

[3]That the Euler and transversality conditions are necessary and sufficient for optimality has been established for more general, convex dynamic optimization problems. See, among others, Stokey and Lucas (1989), Acemoglu (2009). Establishing the necessity of transversality condition for optimality in general has been more challenging; see, Kamihigashi (2001, 2003).

[4]We should clarify at this point that, given an initial output, $y > 0$, a feasible (stochastic) path starting from y (with consumption and investment positive at every date and every realization of the random shock) satisfying the Ramsey-Euler condition is often referred to as a *Ramsey-Euler path*. However, on such a path, only a certain proper subset of all positive output levels might ever be realized. A *Ramsey-Euler policy* , on the other hand, would have to specify consumption for all positive output levels, in such a way that when consumption over time is consistently chosen by following this policy, the path thereby generated is always a Ramsey-Euler path.

[5]For certain versions of our model, in checking for optimality of a Ramsey-Euler path (from an arbitrary initial stock) the transversality condition may be replaced by an infinite number of

sality condition can be a non-trivial task when the stochastic process of output and consumption can reach levels arbitrarily close to zero infinitely often (for instance, on sample paths involving runs of bad realizations of the production shock) and the marginal utility of consumption is infinitely large at zero.

The key contribution of this paper is to develop an alternative sufficient condition for optimality of a Ramsey-Euler policy. Our main result shows that a Ramsey-Euler policy function is optimal if it is continuous. Further, we show that the optimality of the Ramsey-Euler policy function also holds if the consumption and investment policies are both non-decreasing in current output.

As mentioned above, the fact that a policy function satisfies the Ramsey-Euler condition simply ensures that one-period deviation from the path prescribed by the policy function cannot be strictly gainful; it does not, in general, rule out the possibility that deviations from the path for an infinite number of periods may be gainful. The main theoretical insight offered by this paper is that for policy functions that satisfy basic properties such as continuity or monotonicity, satisfying the Ramsey-Euler condition is also sufficient to rule out gainful permanent deviations so that the policy function is optimal.

We establish this result by demonstrating that continuity of the Ramsey-Euler policy ensures that paths generated by this policy (from every initial stock) must satisfy the transversality condition. The implication of this characterization result is that in *applying* the result, one does not need to verify the transversality condition.

It is well known in the literature that in this model, the optimal consumption policy function is unique, continuous, and both the optimal consumption and investment policy functions are non-decreasing (in fact, strictly increasing) in current output.[6] This paper shows that some of these global properties of the (consumption and investment) policy functions that have been shown to be necessary for optimality can also replace the transversality condition in the set of sufficient conditions for the optimality of a Ramsey-Euler policy.

Continuity or monotonicity of the Ramsey-Euler policy can be easily verified for a

"period by period" conditions; see, Brock and Majumdar (1988), Dasgupta and Mitra (1988) and Nyarko (1988). Though such conditions have not been established for the discounted stochastic model considered in this paper, it is worth pointing out that like the transversality condition, these period-by-period conditions taken together involve the entire stochastic process of consumption and capital and establishing optimality by showing that all of them hold can be difficult to implement.

[6] See, for instance, Kamihigashi (2007).

large class of applications of the stochastic growth model (including dynamic games of natural resource use). Our main result allows us to immediately verify optimality of explicit solutions to the Euler equation in certain applications with specific functional forms for the utility and production functions where the policy function is linear. Linearity is however an exception, rather than the rule. As new examples are developed in the future with possibly non-linear Ramsey-Euler consumption functions that may better fit reality, our result will continue to be useful as a way to readily establish optimality. Our main result can also be a useful theoretical tool in proving optimality of a policy function with no explicit form (see remarks at the end of Section 3.2).

Our alternative sufficient condition for optimality of a Ramsey-Euler policy is firmly rooted in the duality approach to the characterization of optimality. A different approach, based on dynamic programming concepts and methods, has also been explored in the literature. Roughly speaking, this method involves guessing the value function from the Ramsey-Euler condition and verifying that this "candidate" value function satisfies the functional equation of dynamic programming, also known as the Bellman equation (see, for instance, Lucas and Stokey, 1989). This approach is useful if the solution to the Bellman equation is unique (for instance, if the utility function is bounded below in the stochastic growth model). Recent advances have extended the applicability of this approach to unbounded utility functions; see, among others, Rincón-Zapatero and Rodriguez-Palmero (2003) and Matkowski and Nowak (2011). In the context of the canonical stochastic optimal growth model, we feel that our result is easier to implement.

The paper is organized as follows. Section 2 outlines the model, the assumptions and some definitions. Section 3 contains the main results of the paper, with Section 3.2 showing that our conditions are necessary and sufficient for optimality. Section 3.3 provides an example to demonstrate that a Ramsey-Euler policy can be discontinuous (and non-monotone), and therefore that a Ramsey-Euler policy is not always optimal.[7] The proof of the main result is contained in Section 4. Section 5 contains an example to illustrate the fact that additional conditions may be needed to ensure optimality of Ramsey-Euler policy in a larger class of stochastic dynamic optimization problems. Section 6 concludes. Section 7 is the appendix and contains proofs of minor results

[7] There is a well-known example of a Ramsey-Euler *path* from an initial condition $y > 0$, which is not an optimal *path* from that initial condition. As far as we know, our example is the first to specify a complete Ramsey-Euler *policy* which generates non-optimal paths from certain initial conditions.

and details of the example in Section 3.3.

2 The Model

We consider an infinite horizon one-good representative agent economy. Time is discrete and is indexed by $t = 0, 1, 2, ...$ At each date $t \geq 0$, the representative agent observes the current stock of output $y_t \in \mathbb{R}_+$ and chooses the level of current investment x_t, and the current consumption level c_t, such that

$$c_t \geq 0, x_t \geq 0, c_t + x_t \leq y_t.$$

This generates y_{t+1}, the output stock next period through the relation

$$y_{t+1} = f(x_t, r_{t+1})$$

where $f(x, r)$ is the production function and r_{t+1} is a random production shock realized at the beginning of period $(t + 1)$.

2.1 Production

We now describe aspects of the above mentioned production process formally. We begin by specifying the nature of the exogenous shocks to production as follows.

(R.1) The sequence of random shocks $\{r_t\}_{t=1}^{\infty}$ is assumed to be an independent and identically distributed random process defined on a probability space $(\Omega, \mathcal{F}, P)$, where the marginal distribution is denoted by μ. The support of this distribution function is a non-empty compact set $I \subset \mathbb{R}$. The distribution function corresponding to μ is denoted by F.

The *production function* is a map f from $\mathbb{R}_+ \times I$ to $\mathbb{R}_+$. We impose the following assumptions[8] on the production function f :

(T.1) Given any $r \in I$, $f(., r)$ is assumed to be continuous, strictly increasing and concave on $\mathbb{R}_+$, with $f(0, r) = 0$, and differentiable on $\mathbb{R}_{++}$, with $f'(\cdot, r) > 0$ on $\mathbb{R}_{++}$. Further, for any $x \geq 0$, $f(x, .) : I \to \mathbb{R}_+$, is a (Borel) measurable function.

[8]Note that we do not require the production function to be monotonic or continuous in the realization of the production shocks.

Define the lower envelope production function $\underline{f}(x) : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\underline{f}(x) = \inf_{r \in I} f(x, r).$$

It is easy to check that $\underline{f}(x)$ is non-decreasing on $\mathbb{R}_+$ and $\underline{f}(0) = 0$. Further, $\underline{f}(x)$ is concave on $\mathbb{R}_+$. It follows that the "worst case" average productivity of investment $[\underline{f}(x)/x]$ is non-increasing in x on $\mathbb{R}_{++}$. The upper envelope production function $\overline{f}(x)$ is defined on $\mathbb{R}_+$ by:

$$\overline{f}(x) = \sup_{r \in I} f(x, r)$$

We assume the following end-point[9] conditions on the production function:

(T.2)

$$\left.\begin{array}{l} (i)\text{ There is } K > 0 \text{ such that } \left[\overline{f}(x)/x\right] < 1 \text{ for all } x > K \\ (ii)\ \lim_{x \downarrow 0}[\underline{f}(x)/x] > 1 \end{array}\right\} \qquad \text{(E)}$$

Note that (E)(ii) implies that the technology is productive near zero for *all* realizations of the production shock. It also implies that $\underline{f}(x) > 0$ for all $x > 0$.

Given an initial stock $y \geq 0$, a stochastic process $\{y_t(y, \omega), c_t(y, \omega), x_t(y, \omega)\}$ is *feasible* from y if it satisfies $y_0 = y$, and:

$$\left.\begin{array}{l} (i)\ c_t(y,\omega) \geq 0, x_t(y,\omega) \geq 0 \text{ for } t \geq 0 \\ (ii)\ c_t(y,\omega) + x_t(y,\omega) \leq y_t(y,\omega),\ y_{t+1}(y,\omega) = f(x_t(y,\omega), r_{t+1}(\omega)) \text{ for } t \geq 0 \end{array}\right\} \qquad \text{(F)}$$

and if for each $t \geq 0$ $\{c_t(y, \omega), x_t(y, \omega)\}$ are $\mathcal{F}_t$ adapted where $\mathcal{F}_t$ is the (sub) σ-field generated by partial history from periods 0 through t.[10]

It is straightforward to verify, using (E)(i), that if $y \geq 0$, and $\{y_t(y, \omega), c_t(y, \omega), x_t(y, \omega)\}$ is feasible from y, then:

$$y_t(y, \omega) \leq K(y), c_t(y, \omega) \leq K(y), x_t(y, \omega) \leq K(y) \text{ for all } t \geq 0 \qquad \text{(B)}$$

where $K(y) = \max\{y, K\}$, and K is given by (E)(i).

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[9] Note that we do not require the production function to satisfy the Inada condition at zero.

[10] We skip formal definitions of sigma fields and sub sigma fields, following a referee's suggestion, as these constructs are standard in the theory of stochastic processes.

2.2 Preferences

Consumption in each period generates an immediate return according to a utility function, $u : \mathbb{R}_{++} \to \mathbb{R}$. The following assumption is imposed on the utility function:

(U.1) u is continuously differentiable, strictly increasing and strictly concave on $\mathbb{R}_{++}$ with $u' > 0$ on $\mathbb{R}_{++}$.

We define

$$u(0) \equiv \lim_{c \downarrow 0} u(c),$$

where the limit is allowed to be finite or $-\infty$.

Finally, the agent discounts future utility using a time invariant discount factor denoted by $\rho \in (0, 1)$.

2.3 The Optimization Problem

Given initial stock $y \geq 0$, the representative agent's objective is to maximize the expected value of the discounted sum of utilities from consumption:

$$E\left[\sum_{t=0}^{\infty} \rho^t u(c_t)\right]$$

subject to feasibility constraints. Under our assumptions, for any feasible stochastic process $\{y_t(y,\omega), c_t(y,\omega), x_t(y,\omega)\}$ from $y \geq 0$, the objective of the representative agent

$$E\left[\sum_{t=0}^{\infty} \rho^t u(c_t(y,\omega))\right]$$

is well defined for any feasible stochastic process $\{y_t(y,\omega), c_t(y,\omega), x_t(y,\omega)\}$ from $y \geq 0$; it is bounded above by $\frac{u(K(y))}{1-\rho}$, but it may equal $-\infty$.

Given initial stock $\overline{y} \geq 0$, a feasible stochastic process $\{y_t(\overline{y},\omega), c_t(\overline{y},\omega), x_t(\overline{y},\omega)\}$ is *optimal* from $\overline{y}$ if for every feasible stochastic process $\{y'_t(\overline{y},\omega), c'_t(\overline{y},\omega), x'_t(\overline{y},\omega)\}$ from $\overline{y}$,

$$E\left[\sum_{t=0}^{\infty} \rho^t u(c_t(\overline{y},\omega))\right] \geq E\left[\sum_{t=0}^{\infty} \rho^t u(c'_t(\overline{y},\omega))\right].$$

2.4 The Optimal Consumption function

A *consumption (policy) function,* is a function $c : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying:

$$0 \leq c(y) \leq y \text{ for all } y \in \mathbb{R}_+$$

Note that this implies $c(0) = 0$. Associated with a consumption function $c(\cdot)$, is an *investment (policy) function* $x : \mathbb{R}_+ \to \mathbb{R}$, defined by

$$x(y) = y - c(y) \text{ for all } y \in \mathbb{R}_+$$

Thus, the investment function $x(\cdot)$ satisfies:

$$0 \leq x(y) \leq y \text{ for all } y \in \mathbb{R}_+$$

A feasible stochastic process $\{y_t(\overline{y}, \omega), c_t(\overline{y}, \omega), x_t(\overline{y}, \omega)\}$ is said to be *generated by* a consumption function $c(y)$ from initial stock $\overline{y} \in \mathbb{R}_+$ if for all $\omega \in \Omega$

$$\begin{aligned} y_0(\overline{y}, \omega) &= \overline{y};\ y_{t+1}(\overline{y}, \omega) = f(y_t(\overline{y}, \omega) - c(y_t(\overline{y}, \omega)), r_{t+1}(\omega)) \text{ for } t \geq 0; \\ c_t(\overline{y}, \omega) &= c(y_t(\overline{y}, \omega)),\ x_t(\overline{y}, \omega) = x(y_t(\overline{y}, \omega)) = y_t(\overline{y}, \omega) - c(y_t(\overline{y}, \omega)) \text{ for } t \geq 0. \end{aligned}$$

A consumption function $c(y)$ is called an *optimal* consumption function if for every $\overline{y} \in \mathbb{R}_+$, the feasible stochastic process $\{y_t(\overline{y}, \omega), c_t(\overline{y}, \omega), x_t(\overline{y}, \omega)\}$ generated by $c(y)$ is optimal from initial stock $\overline{y}$.

3 Main Results

3.1 Some definitions

The main purpose of this paper is to provide a set of tight and easily verifiable sufficient conditions for a consumption (policy) function to be optimal. To this end, we begin with a set of definitions.

A consumption function $c(y)$ is said to be *interior* (or, to satisfy *interiority*) if

$$0 < c(y) < y \text{ for all } y > 0.$$

An *interior* consumption function $c(y)$ is said to satisfy the Ramsey-Euler condition if[11]

$$u'(c(y)) = \rho \int_I u'(c(f(y - c(y), r)))f'(y - c(y), r)dF(r) \text{ for all } y > 0 \tag{RE}$$

In this case we refer to the consumption function $c(y)$ as a *Ramsey-Euler* (consumption) *policy*.

For any *interior* consumption function $c(y)$, the feasible stochastic process $\{y_t(\overline{y}, \omega), c_t(\overline{y}, \omega), x_t(\overline{y}, \omega)\}$ generated by the consumption function $c(y)$ from any initial stock $\overline{y} > 0$ satisfies:

$$y_t(\overline{y}, \omega) > 0, c_t(\overline{y}, \omega) > 0, x_t(\overline{y}, \omega) > 0 \text{ for all } t \geq 0 \text{ and for all } \omega \in \Omega. \tag{P}$$

An interior consumption function $c(y)$ is said to satisfy the *transversality condition* if for all $\overline{y} > 0$:

$$\lim_{t\to\infty} E\{\rho^t u'(c_t(\overline{y}, \omega))x_t(\overline{y}, \omega)\} = 0 \tag{TC}$$

where $\{y_t(\overline{y}, \omega), c_t(\overline{y}, \omega), x_t(\overline{y}, \omega)\}$ is the feasible stochastic process generated by the consumption function $c(y)$ from initial stock $\overline{y}$.

3.2 Main Result: Conditions for Optimality

It is known that if a consumption function is interior, satisfies the Ramsey-Euler condition (RE) and the transversality condition (TC), then it is an optimal consumption function (Mirman and Zilcha 1975). In other words, a Ramsey-Euler policy is optimal if it satisfies the transversality condition (TC). The transversality condition (TC) has also been shown to be necessary for optimality of such a policy. In this section, we outline the main result of the paper: a new set of conditions for optimality of a Ramsey-Euler policy that does not involve the transversality condition (TC). In the proof of this result, we show that under our new optimality conditions, the transversality condition (TC) always holds.

As mentioned in Section 1, the Ramsey-Euler condition is simply a local (first order) condition for optimality that is based on the argument that a *temporary* devi-

[11] For each $y > 0$, the right-hand side of (RE) is ρ times the integral of a non-negative measurable function of r with respect to the distribution function F. It is therefore well defined. (RE) requires that the expression be finite and equal to $u'(c(y))$.

ation from the stochastic process generated by the candidate policy function should not be gainful. It does not rule out the possibility that a *permanent* deviation may be gainful and this possibility is exactly what a condition like the transversality condition (TC) excludes. Verifying the transversality condition can be complicated (though by no means impossible), particularly when the stochastic process of output generated by the candidate consumption function is not necessarily bounded away from zero and the marginal utility at zero is infinite. The complexity can, in fact, be gauged by looking at the proof of our main result where we verify the transversality condition (TC). Without our result, in order to verify rigorously that a RE policy is optimal, one would have to provide a variation of such a proof and the particular variation would depend on the additional information known aboutwhile application of our main result (Theorem 1, formally stated below) to ensure optimality of the consumption policy is indeed trivial. the specific structure of the model.

The following example indicates how even in a setting with specific functional forms and a *linear* Ramsey-Euler consumption function, checking the transversality condition rigorously may not be trivial while application of our main result (Theorem 1, formally stated below) to ensure optimality of the consumption policy is indeed trivial.

Example 1.

Consider the utility and production functions:

$$u(c) = \begin{cases} \frac{c^{1-\eta}}{1-\eta}, \text{ if } c > 0 \\ -\infty, \text{ if } c = 0 \end{cases}$$

$$f(x,r) = \begin{cases} r(x^{1-\eta} + \beta)^{\frac{1}{1-\eta}}, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \end{cases}$$

where

$$\beta > 0, \eta > 1, r \in I = [a,b], 1 < a < b < \infty.$$

Note that $\lim_{x \to 0} f'(x,r) = r \geq a > 1$. Further, assume that

$$\rho E\left(r_t^{-(\eta-1)}\right) < 1 \tag{1}$$

One can easily check that all assumptions in Section 2 are satisfied.[12] It is easy

[12] Benhabib and Rustichini (1994) first specified the deterministic version of this example with

to show that the following linear consumption function satisfies the Ramsey-Euler condition in the stochastic version of this problem:

$$c(y) = \lambda y$$

where

$$\lambda = 1 - \left[\rho E\left(r_t^{-(\eta-1)}\right)\right]^{1/\eta} \in (0,1),$$

using (1).

Observe that

$$\underline{f}(x(y)) = \left[a((1-\lambda)^{1-\eta} + \beta y^{\eta-1})^{\frac{1}{1-\eta}}\right] y < y \text{ for all } y > 0$$

if:

$$a(1-\lambda) < 1$$

i.e., if:

$$\rho < \frac{a^{-\eta}}{E\left(r_t^{-(\eta-1)}\right)} \tag{2}$$

(2) is a stronger condition than (1) which was specified to ensure validity of the Ramsey-Euler consumption function $c(y)$ specified above. Under (2), the stochastic consumption path generated by $c(y)$ is not bounded away from zero (in fact, one can show that consumption enters every neighborhood of zero infinitely often with probability one[13]) and the stochastic process of dual prices (given by the marginal utility of consumption) is unbounded above. Verification of the transversality condition is by no means immediate.[14] ■

Our main result outlined below suggests alternative conditions under which a Ramsey-Euler policy is optimal. Because the transversality condition (TC) is a *necessary* condition of optimality, any alternative sufficient condition for optimality of a Ramsey-Euler policy must ensure the validity of the TC either explicitly or implic-

a linear optimal consumption function. In a dynamic common property resource game, Mitra and Sorger (2014) formally verify the transversality condition for this consumption function.

[13] See, Mitra and Roy (2007), Proposition 2.

[14] To verify the transversality condition directly in this case (i.e., without using the results in our paper) one may use a contraction argument based on the specific functional solution to the Ramsey-Euler condition. This was indicated by an anonymous referee. Such a contraction argument however cannot be readily extended to the general framework considered in this paper.

itly. We will show that continuity, a global property of the consumption function is sufficient to ensure that an interior Ramsey-Euler consumption function is optimal. In itself, this does not require us to verify any monotonicity property of the consumption function. However, from Brock and Mirman (1972, Lemma 1.1, p. 489), we know that optimal consumption and investment must be strictly increasing in output. Thus, it must be the case that continuity of a Ramsey-Euler policy implies that both the consumption and investment functions are strictly increasing. It turns out that the converse is also true; continuity of a consumption function is also implied by the monotonicity of the consumption and investment functions. It is a simple and useful result, which we have not found explicitly mentioned in the literature and is stated below as a lemma.

Lemma 1 *If $c(y)$ is a consumption function such that $c(y)$ and $x(y) = y - c(y)$ are non-decreasing on $\mathbb{R}_+$, then $c(y)$ is continuous on $\mathbb{R}_+$.*

The proof of Lemma 1 is contained in the Appendix. Note that Lemma 1 applies to any consumption function including ones that do not satisfy interiority or the Ramsey-Euler condition. Lemma 1 allows us to establish an equivalence between continuity and monotonicity properties of a Ramsey-Euler consumption function.

We are now ready to state the main result:

Theorem 1 *Suppose that $c(\cdot)$ is an interior consumption function. Then the following statements are equivalent:*

(a) $c(y)$ is continuous and satisfies the Ramsey-Euler condition (RE)

(b) $c(y)$ and $y - c(y)$ are nondecreasing on $\mathbb{R}_+$ and $c(y)$ satisfies the Ramsey-Euler condition (RE)

(c) $c(y)$ and $y - c(y)$ are strictly increasing on $\mathbb{R}_+$ and $c(y)$ satisfies the Ramsey-Euler condition (RE)

(d) $c(y)$ is optimal

The proof of Theorem 1 is contained in Section 4.

In the literature on the one sector stochastic growth model, it is well known that an interior optimal consumption function must satisfy the Ramsey-Euler condition (see, for instance, Theorem 1 in Mirman and Zilcha 1975). Under the assumptions imposed in our model, there exists a unique optimal consumption function. If this consumption function is interior, then both optimal consumption and optimal investment are continuous and strictly increasing in output (see, for instance, Theorem 2.1 in Kamihigashi 2007).[15] The contribution of Theorem 1 is to provide a new set of the sufficient conditions for optimality; a consumption function is optimal if it is interior, satisfies the Ramsey-Euler condition (RE) and is either continuous or (equivalently), consumption and investment are nondecreasing in output.

It should be noted that an optimal consumption function is necessarily interior if the utility function satisfies the Inada condition at zero i.e., $u'(c) \to +\infty$ as $c \to 0$, and in that case the statement of Theorem 1 can be appropriately modified to include interiority as one of the conditions that are necessary and sufficient for optimality of any consumption function.

Theorem 1 shows that for a policy function satisfying the Ramsey-Euler condition, which is a functional equation, continuity (or monotonicity) of the function is necessary and sufficient for the policy to be optimal. In doing so, it offers an exact characterization of when the stochastic process generated by such a policy function satisfies the transversality condition. This equivalence between the transversality property of the stochastic path generated by a Ramsey-Euler policy and certain global properties of the policy function is a basic theoretical insight. To the best of our knowledge, no such equivalence result has been established in the existing literature and indicates that it may be possible to develop alternative optimality conditions in a wider class of stochastic dynamic models.

Theorem 1 can be useful for practitioners who analyze the stochastic optimal growth model (or a dynamic optimization problem of similar structure, for instance in certain dynamic games of natural resource exploitation[16]) with specific functional forms for the utility and production functions and where a consumption function with an explicit functional form can be shown to satisfy the Ramsey-Euler condition. It allows us to verify optimality of this policy function without having to analyze the asymptotic behavior of the present value of stochastic investment paths generated by the policy function and the stochastic production technology. With an explicit form for the consumption function, it is easy to directly check whether it is continuous or monotonic in output.

[15] While some of these results are usually established under the assumption that the utility function satisfies the Inada condition at zero (i.e., $u'(c) \to +\infty$ as $c \to 0$) so that the optimal consumption function is interior, the actual proofs only use interiority.

[16] See, among others, Levhari and Mirman (1980), Mitra and Sorger (2014).

For instance, the interior Ramsey-Euler consumption function outlined in Example 1 is linear and hence continuous, so that Theorem 1 immediately shows that it is optimal without having to go through the task of verifying the transversality condition.[17]

However, Theorem 1 can also be useful in proving theoretical results when there is no explicit functional form for the "candidate" policy function. Depending on the economic problem being addressed, such a consumption function may be obtained from some other optimization problem or as the equilibrium of some other economic model, and the structure of these other problems may ensure continuity or monotonicity of this function.

For instance, consider finite horizon versions of the optimization problem in our model. Under certain standard assumptions, there exists a unique interior (initial period) optimal consumption function in such a finite horizon model. Now, consider a "candidate" consumption function that is defined as the point-wise limit of these optimal consumption functions (as the time horizon tends to infinity). Given that this is a dynamic optimization problem with unbounded immediate reward (utility), showing that the limit policy function is optimal in the infinite horizon case gets to be somewhat complicated. It is, however, easy to show that these finite horizon optimal consumption functions are continuous and monotonic (in the sense that both consumption and investment are non-decreasing in current output), that they satisfy the (initial period) Ramsey-Euler condition for the finite horizon problem, and that they are weakly decreasing over time. Taking appropriate limits, one can show that the limit consumption function satisfies the Ramsey-Euler condition for the infinite horizon problem and that consumption and investment are non-decreasing in output. Theorem 1 can now be directly applied to establish optimality of the limit

[17] As another simple application of our result, consider the well-known example, in which:

$$f(x, r) = rx^{1-\alpha} \text{ for all } x \geq 0 \text{ and for all } r \in I \equiv [a, b]$$

with $0 < a < b < \infty$, and $\alpha \in (0, 1)$, and:

$$u(c) = \ln c \text{ for all } c > 0$$

In this case, the consumption policy function:

$$c(y) = [1 - \rho(1 - \alpha)]y \text{ for all } y \geq 0$$

is a Ramsey-Euler policy. Since it is clearly continuous, it is an optimal policy by Theorem 1. No other condition needs to be checked to arrive at this conclusion.

consumption function.[18]

3.3 Non-Optimal Ramsey-Euler Policy: An Example

Our main result on sufficiency in Section 3.2 (Theorem 1) indicates that an interior consumption function satisfying the Ramsey-Euler condition is optimal if it is continuous. An important question that arises then is whether it is at all possible for an interior consumption function to be discontinuous and still satisfy the Ramsey -Euler condition. Of course, if there is such a consumption function it would not be optimal because (as indicated in the previous subsection) every optimal consumption must be continuous. On the other hand, in the absence of any such example, one may have reason to be sceptical about the tightness of our sufficiency result; in particular, if every Ramsey-Euler policy function is automatically continuous, then continuity would be superfluous in the statement of our sufficient condition for optimality (just as transversality condition is superfluous, given any continuous Ramsey-Euler policy). The example outlined below shows that this is *not* the case; there are Ramsey-Euler policies that are discontinuous.

In particular, we provide an example of an economy and an interior consumption function that satisfies the Ramsey-Euler condition (RE) but violates continuity; as one would expect (for instance, on the basis of Theorem 1), it is not an optimal consumption function. This consumption function is also non-monotonic. The technology in this example is deterministic i.e., the production function is independent of random shocks (or equivalently, the random shocks have a degenerate distribution).[19]

Example 2.

The deterministic production function $f(x)$, the utility function $u(c)$ and the discount factor ρ are specified as follows:

$$\left.\begin{array}{l} (i)\ f(x) = 2x^{1/2} \ \text{ for all } x \in [0,4];\ f(x) = 2 + (1/2)x \text{ for all } x > 4 \\ (ii)\ u(c) = \ln c \ \text{ for all } c \in \mathbb{R}_{++};\ u(0) \equiv \lim_{c \downarrow 0} u(c) = -\infty \\ (iii)\ \rho \in (0,1) \end{array}\right\} \qquad \text{(EX)}$$

Note that (f, u, ρ) specified in (EX) satisfy all assumptions in Section 2. Assumption (**T.2**)(i) is satisfied by choosing (for instance) $K = 4$ and (**T.2**)(ii) is satisfied as

[18] See, Mitra and Roy (2017), proof of Lemma 10, for such an application of Theorem 1.

[19] The deterministic framework is a special case of the model described in Section 2, with I specified to be a singleton set, and is, in fact, a version of the well-known Ramsey-Cass-Koopmans model.

$[f(x)/x] \to \infty$ as $x \to 0$.

Define a consumption function $c(y)$ by:

$$c(y) = \begin{cases} (1-(\rho/2))y & \text{for all } 0 \leq y \leq 4 \\ (1-\rho)(y-4) & \text{for all } y > 4 \end{cases} \tag{3}$$

Note that c is both discontinuous and non-monotone at $y = 4$.

Define the sets A, B by:

$$A = (0, 4], B = (4, \infty).$$

It is easy to check the following: (a) c is an interior consumption function, (b) if the initial stock $y \in A$, then the sequence $\{y_t, c_t, x_t\}$ generated by c has the property that $y_t \in A$ for all $t \geq 0$, and (c) if the initial stock $y \in B$, then the sequence $\{y_t, c_t, x_t\}$ generated by c has the property that $y_t \in B$ for all $t \geq 0$. The details of (a)-(c) are contained in the Appendix. One implication of (b) and (c) is that in verifying the Ramsey-Euler equation (RE), one is never switching between the consumption functions given in the first and second lines of (3).

For the deterministic economy in this example, the Ramsey-Euler (RE) condition reduces to:

$$u'(c(y)) = \rho u'(c(f(y - c(y)))f'(x(y)) \text{ for all } y > 0. \tag{4}$$

One can check using (3),

$$u'(c(y)) = \frac{1}{(1-(\rho/2))y} = \rho u'(c(f(y - c(y)))f'(x(y)) \text{ for any } y \in A$$

and

$$u'(c(y)) = \frac{1}{(1-\rho)(y-4)} = \rho u'(c(f(y - c(y)))f'(x(y)) \text{ for any } y \in B$$

so that c satisfies the Ramsey-Euler condition (4).

We now verify directly (i.e., without using Theorem 1) that c is *not* an optimal consumption function. To see this, choose $y = 5 \in B$. Then, c generates the sequence $\{y_t, c_t, x_t\}$ where $y_t \in B$ and, in particular, $y_t > 4$ for all $t \geq 0$. Further, $x(y) = y - (1-\rho) > 4$ so that $f(x(y)) < \rho y + (1-\rho)4 < y$. This implies that the output sequence $\{y_t\}$ satisfies:

$$y_{t+1} < y_t \text{ and } y_t > 4 \text{ for all } t \geq 0 \tag{5}$$

and the corresponding consumption sequence $\{c_t\}$ satisfies:

$$c_t = c(y_t) = (1-\rho)(y_t - 4) \leq (1-\rho)(y-4) = (1-\rho) < 1 \text{ for all } t \geq 0 \tag{6}$$

However, the sequence $\{y'_t, c'_t, x'_t\}$ from $y = 5$, given by:

$$x'_t = 1 \text{ for all } t \geq 0; y'_0 = 5, y'_t = 2 \text{ for all } t \geq 1; c'_0 = 4, c'_t = 1 \text{ for all } t \geq 1 \tag{7}$$

is clearly feasible from $y = 5$. Notice that $c'_t > c_t$ for all $t \geq 0$. Thus, the sequence $\{y_t, c_t, x_t\}$ is not optimal from $y = 5$, and consequently c is not an optimal consumption function. In the Appendix, we verify explicitly (i.e., without using the standard characterization result of optimality given in Mirman and Zilcha (1975)) that the sequence $\{y_t, c_t, x_t\}$ from $y = 5$ does not satisfy the transversality condition[20]. This concludes the example. ■

4 Proof of Main Result

This section contains the proof of Theorem 1 which is the main result in this paper. The proof is based on a more general lemma stated below.

Lemma 2 *Suppose $c(\cdot)$ is a consumption function satisfying the following conditions:*

(i) $c(y)$ is interior

(ii) $x(y) = y - c(y)$ is non-decreasing on $\mathbb{R}_+$

(iii) for every y_1, y_2 satisfying $0 < y_1 \leq y_2$,

$$\inf\{c(y) : y \in [y_1, y_2]\} > 0$$

(iv) $c(y)$ satisfies the Ramsey-Euler condition (RE).

Then $c(\cdot)$ is an optimal consumption function.

[20]This is in response to a suggestion by an anonymous referee.

Lemma 2 indicates that an interior consumption function that satisfies the Ramsey-Euler equation is optimal as long as the associated investment function is non-decreasing and, in addition, consumption on every strictly positive closed interval is bounded away from zero. If these conditions are satisfied, then the transversality condition (TC) is automatically satisfied, so that the candidate consumption function is optimal. Before coming to the proof of Lemma 2, we first indicate how the proof of Theorem 1 follows from Lemma 2.

Proof of Theorem 1:

As $c(y)$ is continuous on $\mathbb{R}_+$, for any y_1, y_2 satisfying $0 < y_1 \leq y_2$, there exists $z \in [y_{1,}y_2]$ such that $c(z) = \inf\{c(y) : y \in [y_{1,}y_2]\}$ and using interiority of $c(y)$,we have that $c(z) > 0$ so that condition (iii) of Lemma 2 is satisfied. Next, we show that $x(y) = y - c(y)$ is non-decreasing on $\mathbb{R}_+$. To see this, suppose to the contrary that there exists $y' > y \geq 0$ such that $0 \leq x(y') < x(y)$. Since $x(0) = 0$, we must have $y > 0$. As $c(.)$ is interior, we must have $z \equiv x(y') > 0$. As $c(.)$ is continuous on $\mathbb{R}_+$, so is $x(.)$. Since $x(0) = 0 < z = x(y') < x(y)$, we can use the intermediate value theorem to find $y'' \in (0, y)$, such that $x(y'') = z \equiv x(y')$. Now using the Ramsey-Euler equation (RE) for y' and y'' and noting that $y' - c(y') = x(y')$, $y'' - c(y'') = x(y'')$ we get

$$u'(c(y')) = \rho \int_I u'(c(f(x(y'), r)))f'(x(y'), r)dF(r) \tag{8}$$

$$u'(c(y'')) = \rho \int_I u'(c(f(x(y''), r)))f'(x(y''), r)dF(r) \tag{9}$$

As $x(y') = x(y'')$, the right hand expressions in (8) and (9) are equal so that

$$u'(c(y')) = u'(c(y'')) \tag{10}$$

But $y'' < y < y'$ and $x(y') = x(y'')$ implies that

$$c(y') = y' - x(y') > y'' - x(y'') = c(y'')$$

which contradicts (10) since u is strictly concave on $\mathbb{R}_{++}$. Thus, $x(y) = y - c(y)$ is non-decreasing on $\mathbb{R}_+$. We have now verified that conditions (i)-(iv) of Lemma 2 hold. Theorem 1 follows. ■

We now turn to the proof of Lemma 2. We use duality theory to establish the lemma. From the stochastic path (of consumption, output and investment) generated by the Ramsey-Euler consumption function, we define the stochastic process of dual prices (where the price in each period equals the present value of the marginal utility of consumption). The proof proceeds through the following steps:

1. Establish competitive properties of the consumption and investment path
2. Show that the Transversality Condition holds
3. Establish optimality

Steps 1 and 3 rely on existing arguments. The key innovation in the proof is in Step 2 where we show that the transversality condition (TC) necessarily holds. In other words, to *prove* Lemma 2, we verify the transversality condition, so that in *applying* the lemma, one does not have to. Verifying the transversality condition essentially reduces to showing that the expected value of output converges to zero in the long run.

The difficulty in the proof arises when the consumption function is such that in a neighborhood of zero, output may decline for "bad" realizations of the shock; as a result the stochastic process of output, and hence consumption, may be arbitrarily close to zero. In the latter case (Case (ii) of Step 2 in the proof), there are possible sample paths along which prices and therefore the realized value of output may be bounded away from zero and may even diverge to infinity. The proof proceeds to show that despite all this, the *expectation* of the value of output converges to zero. This is achieved by showing in three sub-steps that:

a. If there are sample paths of output that get arbitrarily close to zero, then (as the technology allows for growth with certainty near zero) there must be serious under-investment when output is small and in particular, the propensity to consume must be bounded away from zero.

b. The *expected prices* are uniformly bounded over time even if the sample paths get close to zero; this is achieved by comparing the expected discounted sum of utility generated by the consumption function to a specifically chosen comparison feasible path where consumption is bounded away from zero

c. The expected value of output (or investment) converges to zero.

In making these arguments, conditions (i)-(iii) of the lemma are used to ensure that consumption is bounded away from zero (prices are bounded above) not only in every period but also uniformly over time on sample paths where the output over time is uniformly bounded away from zero.

Proof of Lemma 2.

Let $Y = \mathbb{R}_+$. Fix initial stock $\overline{y} \in Y$ with $\overline{y} > 0$. Consider the stochastic process of output, consumption and investment $\{y_t(\overline{y}, \omega), c_t(\overline{y}, \omega), x_t(\overline{y}, \omega)\}_{t=0}^{\infty}$ for $\omega \in \Omega$, hereafter written as $\{\mathbf{y}_t, \mathbf{c}_t, \mathbf{x}_t\}$, generated by the consumption function $c(y)$.[21] Using condition (i) of the lemma, $\mathbf{y}_t > 0, \mathbf{c}_t > 0, \mathbf{x}_t > 0$ for all $t \geq 0$. Equality or inequalities involving these random variables should be interpreted as holding for all $\omega \in \Omega$. Note that $\{\mathbf{y}_t, \mathbf{c}_t, \mathbf{x}_t\}$ is *feasible* from y. We have to establish that it is *optimal* from y.

Step 1: Duality and "competitive" properties of the Ramsey-Euler path

Let $\{\underline{y}_t\}$ be the deterministic sequence defined by:

$$\underline{y}_0 = \overline{y}, \underline{y}_{t+1} = \underline{f}(x(\underline{y}_t)), t \geq 0. \tag{11}$$

Under (**T.1**) $\underline{f}(.)$ is nondecreasing on Y. Under condition (ii) of the lemma, $x(.)$ is nondecreasing on Y. It is therefore easy to check that for all $t \geq 0$:

$$K(\overline{y}) \geq \mathbf{y}_t \geq \underline{y}_t. \tag{12}$$

As $x(z) > 0$ for all $z > 0$ and (using $(T.2)$) $\underline{f}(x) > 0$ for all $x > 0$,

$$\underline{y}_t > 0 \text{ for all } t \geq 0. \tag{13}$$

Let $\{\underline{c}_t\}$ be the sequence defined by:

$$\underline{c}_t = \inf\{c(z) : z \in [\underline{y}_t, K(\overline{y})]\} \text{ for all } t \geq 0. \tag{14}$$

Using (13) and condition (iii) of the lemma, $\underline{c}_t > 0$ for all $t \geq 0$. Further, using (12) and (14), we have:

$$\mathbf{c}_t = c(\mathbf{y}_t) \geq \underline{c}_t > 0 \text{ for all } t \geq 0. \tag{15}$$

Thus, for every $t \geq 0$:

$$-\infty < u(\underline{c}_t) \leq u(c_t) \leq u(K(\overline{y})). \tag{16}$$

[21] This notational simplification follows a suggestion by two anonymous referees.

so that for each t, $u(\mathbf{c}_t)$ is a bounded $\mathcal{F}_t-$measurable function and has finite expectation.

Using (15), we can define the stochastic price process $\{p_t(\overline{y},\omega)\}$, hereafter written as $\{\mathbf{p}_t\}$, by:

$$\mathbf{p}_t = \rho^t u'(\mathbf{c}_t)) \text{ for } t \geq 0. \tag{17}$$

As before, equality or inequalities involving these random variables should be interpreted as holding for all $\omega \in \Omega$. It follows (from (15)) that for every $t \geq 0$,

$$\mathbf{p}_t \leq \rho^t u'(\underline{c}_t) \ < \infty$$

i.e., $\mathbf{p}_t$ is a bounded $\mathcal{F}_t-$measurable random variable (and hence integrable) for each t.

For all $c \geq 0$, and all $t \geq 0$, we have by concavity of u and (17),

$$\rho^t u(\mathbf{c}_t) - \mathbf{p}_t\mathbf{c}_t \geq \rho^t u(c) - \mathbf{p}_t c \tag{18}$$

so that for each $t \geq 0$, we have:

$$E\rho^t u(\mathbf{c}_t) - E\mathbf{p}_t\mathbf{c}_t \geq E\rho^t u(\tilde{\mathbf{c}}_t) - E\mathbf{p}_t\tilde{\mathbf{c}}_t \tag{19}$$

for every bounded $\mathcal{F}_t$ measurable random variable $\tilde{\mathbf{c}}_t \geq 0$ defined on Ω. Note that (using (16)), $E\rho^t u(\mathbf{c}_t)$ is finite; further, as $\tilde{\mathbf{c}}_t$ is bounded, $E\rho^t u(\tilde{\mathbf{c}}_t)$ on the right hand side of (19) is well defined though it may be $-\infty$.

Using the Ramsey-Euler condition (RE) and (17), one can see that[22]:

$$\mathbf{p}_t = \rho^t u'(\mathbf{c}_t)) = E\{\mathbf{p}_{t+1} f'(\mathbf{x}_t, r_{t+1})|\mathcal{F}_t\} \tag{20}$$

Using the concavity of f (in x) we have for all $x \geq 0$ and all $t \geq 0$,

$$f(x, r_{t+1}) - f(\mathbf{x}_t, r_{t+1}) \leq f'(\mathbf{x}_t, r_{t+1})(x - \mathbf{x}_t)$$

so that:

$$\mathbf{p}_{t+1} f(x, r_{t+1}) - \mathbf{p}_{t+1} f(\mathbf{x}_t, r_{t+1}) \leq \mathbf{p}_{t+1} f'(\mathbf{x}_t, r_{t+1})(x - \mathbf{x}_t) \tag{21}$$

[22]Strictly speaking, this involves switching from conditional expectation with respect to the distribution function F to a conditional expectation with respect to a sub sigma field. Following the suggestion made by an anonymous referee, we skip the proof as it is standard.

Thus, for every bounded $\mathcal{F}_t$ measurable random variable $\tilde{\mathbf{x}}_t \geq 0$ defined on Ω, taking the conditional expectation with respect to $\mathcal{F}_t$ in (21) with $x = \tilde{\mathbf{x}}_t$ we get:

$$\begin{aligned} & E\{\mathbf{p}_{t+1} f(\tilde{\mathbf{x}}_t, r_{t+1}) | \mathcal{F}_t\} - E\{\mathbf{p}_{t+1} f(\mathbf{x}_t, r_{t+1}) | \mathcal{F}_t\} \\ \leq \; & E\{\mathbf{p}_{t+1} f'(\mathbf{x}_t, r_{t+1})(\tilde{\mathbf{x}}_t - \mathbf{x}_t) | \mathcal{F}_t\} \\ = \; & (\tilde{\mathbf{x}}_t - \mathbf{x}_t) E\{\mathbf{p}_{t+1} f'(\mathbf{x}_t, r_{t+1}) | \mathcal{F}_t\} = \mathbf{p}_t(\tilde{\mathbf{x}}_t - \mathbf{x}_t) \end{aligned} \tag{22}$$

where the third line uses the fact that $\tilde{\mathbf{x}}_t$ and $\mathbf{x}_t$ are $\mathcal{F}_t$ measurable and the last line in (22) uses (20). Transposing terms in (22), for every bounded $\mathcal{F}_t$ measurable $\tilde{\mathbf{x}}_t \geq 0$, we have:

$$E\{\mathbf{p}_{t+1} f(\mathbf{x}_t, r_{t+1}) | \mathcal{F}_t\} - \mathbf{p}_t \mathbf{x}_t \geq E\{\mathbf{p}_{t+1} f(\tilde{\mathbf{x}}_t, r_{t+1}) | \mathcal{F}_t\} - \mathbf{p}_t \tilde{\mathbf{x}}_t \tag{23}$$

so that:

$$E\{\mathbf{p}_{t+1} f(\mathbf{x}_t, r_{t+1})\} - E\{\mathbf{p}_t \mathbf{x}_t\} \geq E\{\mathbf{p}_{t+1} f(\tilde{\mathbf{x}}_t, r_{t+1})\} - E\{\mathbf{p}_t \tilde{\mathbf{x}}_t\} \tag{24}$$

Next, one can show that for *any feasible* stochastic process of output, consumption and investment $\{\widetilde{\mathbf{y}}_t, \widetilde{\mathbf{c}}_t, \widetilde{\mathbf{x}}_t\}$ from initial stock $\overline{y}$, and for every $T \in \mathbb{N}$

$$E\{\sum_{t=0}^{T} \rho^t u(\widetilde{\mathbf{c}}_t)\} - E\{\sum_{t=0}^{T} \rho^t u(\mathbf{c}_t)\} \leq E\{\mathbf{p}_T \mathbf{x}_T\} - E\{\mathbf{p}_T \widetilde{\mathbf{x}}_T\} \tag{25}$$

To see (25), note that from (19) we have for $t \geq 1$

$$\begin{aligned} & E\rho^t u(\widetilde{\mathbf{c}}_t) - E\rho^t u(\mathbf{c}_t) \\ \leq \; & E\mathbf{p}_t \widetilde{\mathbf{c}}_t - E\mathbf{p}_t \mathbf{c}_t = [E\mathbf{p}_t \widetilde{\mathbf{y}}_t - E\mathbf{p}_t \widetilde{\mathbf{x}}_t] - [E\mathbf{p}_t \mathbf{y}_t - E\mathbf{p}_t \mathbf{x}_t] \\ = \; & [E\mathbf{p}_t \widetilde{\mathbf{y}}_t - E\mathbf{p}_{t-1} \widetilde{\mathbf{x}}_{t-1}] + [E\mathbf{p}_{t-1} \widetilde{\mathbf{x}}_{t-1} - E\mathbf{p}_t \widetilde{\mathbf{x}}_t] \\ & -[E\mathbf{p}_t \mathbf{y}_t - E\mathbf{p}_{t-1} \mathbf{x}_{t-1}] - [E\mathbf{p}_{t-1} \mathbf{x}_{t-1} - E\mathbf{p}_t \mathbf{x}_t] \\ \leq \; & [E\mathbf{p}_{t-1} \widetilde{\mathbf{x}}_{t-1} - E\mathbf{p}_t \widetilde{\mathbf{x}}_t] - [E\mathbf{p}_{t-1} \mathbf{x}_{t-1} - E\mathbf{p}_t \mathbf{x}_t] \end{aligned}$$

where the first inequality uses (19) and the second inequality uses (24).

Step 2: Transversality Condition

Next, we come to the main step in the proof where we show that the transversality condition:

$$\lim_{t\to\infty} E\{\mathbf{p}_t\mathbf{x}_t\} = 0 \tag{26}$$

holds. We separate our analysis into two cases:

(i) there is a sequence $\{y^j\}_{j=1}^{\infty}$, with $y^j > 0$, and $y^j \downarrow 0$ as $j \uparrow \infty$, such that $[\underline{f}(x(y^j))/y^j] \geq 1$ for all $j \in \mathbb{N}$;

(ii) there is $\theta \in (0, K)$ such that $[\underline{f}(x(y'))/y'] < 1$ for all $y' \in J \equiv (0, \theta)$.

Case (i):

Since $y_0 = \overline{y} > 0$, we can find $n \in \mathbb{N}$, such that $\overline{y} > y^n$. Fix n. Then, as $x(.)$ is non-decreasing (condition (ii) of the lemma),

$$\mathbf{y}_1 = f(x(\mathbf{y}_0), r_1) \geq \underline{f}(x(\mathbf{y}_0)) \geq \underline{f}(x(y^n)) = \frac{\underline{f}(x(y^n))}{y^n} y^n \geq y^n.$$

By induction, one can then show that $\mathbf{y}_t \in [y^n, K(\overline{y})]$ for all $t \geq 0$.

Define:

$$m = \inf\{c(z) : z \in [y^n, K(\overline{y})]\}.$$

As $y^n > 0$, using condition (iii) of the lemma, we have $m > 0$. For all $t \geq 0$, $\mathbf{c}_t = c(\mathbf{y}_t) \geq m$ so that:

$$E\{\mathbf{p}_t\mathbf{y}_t\} \leq E\{\rho^t u'(\mathbf{c}_t)K(\overline{y})\} \leq \rho^t u'(m))K(\overline{y})$$

and (26) is clearly satisfied.

Case (ii): This is the more difficult case and our proof here proceeds through several sub-steps.

Sub-step a. A Lower Bound on the Propensity to Consume

Letting $y' \downarrow 0$, we have $x(y') \downarrow 0$. Using the fact that (by (**T.2**)(ii)) $\lim_{x\downarrow 0} \left[\underline{f}(x)/x\right] > 1$, there exists $\eta > 0$ and $\theta' \in (0, \theta)$ such that,

$$\left[\underline{f}(x(y'))/x(y')\right] \geq 1 + \eta \text{ for all } y' \in J' = (0, \theta']. \tag{27}$$

Using (27), we have for all $y' \in J'$,

$$1 > \left[\underline{f}(x(y'))/y'\right] = \left[\underline{f}(x(y'))/x(y')\right] [x(y')/y'] \geq (1 + \eta)[x(y')/y'] \tag{28}$$

so that for all $y' \in J'$,

$$[x(y')/y'] \leq [1/(1+\eta)] < 1 \tag{29}$$

and consequently, for all $y' \in J'$,

$$[c(y')/y'] \geq [\eta/(1+\eta)] > 0 \tag{30}$$

Define

$$M = \inf\{c(z) : z \in [\theta', K(\overline{y})]\}$$

As $\theta' > 0$, using condition (iii) of the lemma, $M > 0$. For $y' \in (\theta', K(y)]$, we have

$$c(y')/y' \geq M/K\,(\overline{y}) \tag{31}$$

Using (30) and (31) and noting that $0 < \theta' < \theta < K(\overline{y})$,

$$[c(y')/y'] \geq \min\{\eta/(1+\eta), M/K(\overline{y})\} \equiv \zeta > 0 \text{ for all } y' \in (0, K(\overline{y})] \tag{32}$$

Thus, in case (ii), the propensity to consume is bounded away from zero on $(0, K(\overline{y})]$.

Sub-step b. An Upper Bound on the Expected Price

Define:

$$\delta = \min\{\overline{y}/(1+\eta), x(\theta')\} \tag{33}$$

Now, define for $t \geq 0$,

$$A(t) = \{\omega \in \Omega : \mathbf{y}_t < (\delta/2)\}; D(t) = \{\omega \in \Omega : \mathbf{y}_t \geq (\delta/2)\} \tag{34}$$

Notice that on $D(t)$, consumption will have a positive lower bound (by Step 1), and so the price will be bounded above. On the "problematic" set $A(t)$, consumption can be very small for some realizations of the random shock, leading to very high prices for those realizations. The important result at this stage is that, nevertheless, the integral of the prices on this set is uniformly bounded above. This result is obtained by defining a comparison path, on which the input level is fixed at δ, so that consumption absorbs all the effect of the random shocks.

Note that, by (33), $\delta \leq \overline{y}/(1+\eta)$ and so $\overline{y} - \delta \geq \eta\delta$. Also, $0 < \delta \leq x(\theta')$, and so:

$$[\underline{f}(\delta)/\delta] \geq [\underline{f}(x(\theta'))/x(\theta')] \geq 1 + \eta. \tag{35}$$

where the second inequality in (35) follows from (27).

Now, define the stochastic process of output, consumption and investment $\{\mathbf{y}'_t, \mathbf{c}'_t, \mathbf{x}'_t\}$ as follows:

$$\begin{aligned} \mathbf{y}'_0 &= \mathbf{y}_0 = \overline{y}, \mathbf{x}'_t = \delta,\ \mathbf{y}'_{t+1} = f(\mathbf{x}'_t, r_{t+1}) \text{ for } t \geq 0 \\ \mathbf{c}'_t &= \mathbf{y}'_t - \mathbf{x}'_t \text{ for } t \geq 0 \end{aligned} \tag{36}$$

It is straightforward to verify that $\{\mathbf{y}'_t, \mathbf{c}'_t, \mathbf{x}'_t\}$ is feasible from initial stock $\overline{y}$ and:

$$\mathbf{c}'_t \geq \eta\delta > 0 \text{ for all } t \geq 0 \tag{37}$$

Further, using (25) and (37),

$$\begin{aligned} & E\{\mathbf{p}_T \mathbf{x}'_T\} - E\{\mathbf{p}_T \mathbf{x}_T\} \\ \leq\ & E\{\sum_{t=0}^{T} \rho^t u(\mathbf{c}_t)\} - E\{\sum_{t=0}^{T} \rho^t u(\mathbf{c}'_t)\} \\ \leq\ & E\{\sum_{t=0}^{T} \rho^t u(K(\overline{y}))\} - E\{\sum_{t=0}^{T} \rho^t u(\eta\delta)\} \\ \leq\ & \frac{[u(K(\overline{y})) - u(\eta\delta)]}{\rho} \equiv Q \end{aligned} \tag{38}$$

Recalling the definitions of $A(t)$ and $D(t)$ for $t \geq 0$, we can write:

$$E[\mathbf{p}_t\{\mathbf{x}'_t - \mathbf{x}_t\}] = \int_{A(t)} [\mathbf{p}_t\{\mathbf{x}'_t - \mathbf{x}_t\}]\, P(d\omega) + \int_{D(t)} [\mathbf{p}_t\{\mathbf{x}'_t - \mathbf{x}_t\}] P(d\omega) \tag{39}$$

The first integral on the right hand side of (39) can then be evaluated as:

$$\int_{A(t)} [\mathbf{p}_t\{\mathbf{x}'_t - \mathbf{x}_t\}]\, P(d\omega) \geq (\delta/2) \int_{A(t)} \mathbf{p}_t P(d\omega) \tag{40}$$

since $\mathbf{x}'_t = \delta$ for all $t \geq 0$ while $\mathbf{x}_t < (\delta/2)$ on $A(t)$ by (34). The second integral on the right hand side of (39) can be evaluated as:

$$\begin{aligned} \int_{D(t)} [\mathbf{p}_t\{\mathbf{x}'_t - \mathbf{x}_t\}] P(d\omega) &\geq -\int_{D(t)} \{\mathbf{p}_t \mathbf{x}_t)\} P(d\omega) \\ &\geq -\rho^t u'(\zeta(\delta/2)) K(\overline{y}) \geq -u'(\zeta(\delta/2)) K(\overline{y}) \end{aligned} \tag{41}$$

since on $D(t)$, we have by (32), $\mathbf{c}_t \geq \zeta \mathbf{y}_t \geq \zeta(\delta/2)$, and so $\mathbf{p}_t = \rho^t \ u'(\mathbf{c}_t) \leq \rho^t u'((\zeta\delta/2))$, while $\mathbf{x}_t \leq K(\overline{y})$. Using (40) and (41) in (39), we obtain:

$$E[\mathbf{p}_t\{\mathbf{x}_t' - \mathbf{x}_t\}] \geq (\delta/2) \int_{A(t)} \mathbf{p}_t P(d\omega) - u'(\zeta(\delta/2))K(\overline{y})] \tag{42}$$

Using (42) in (38), we get for all $T \in \mathbb{N}$,

$$(\delta/2) \int_{A(T)} \mathbf{p}_T P(d\omega) \leq Q + u'(\zeta(\delta/2))K(\overline{y}) \equiv Q' \tag{43}$$

which can be rewritten as:

$$\int_{A(T)} \mathbf{p}_T P(d\omega) \leq [2Q'/\delta] \text{ for all } T \in \mathbb{N} \tag{44}$$

Sub-step c. Establishing the Transversality Condition

We now proceed to establish (26) as follows. Given any $\varepsilon > 0$, we show that there is $\overline{N} \in \mathbb{N}$ such that:

$$E\{\mathbf{p}_T \mathbf{x}_T\} \leq \varepsilon \text{ for all } T \geq \overline{N} \tag{45}$$

To this end, given any $\varepsilon > 0$, we first choose $N \in \mathbb{N}$ with $N > 1$, such that:

$$2Q'/N < (\varepsilon/2) \tag{46}$$

where Q' is defined in (43), and Q in (38). Next, we choose $\overline{N} \in \mathbb{N}$, such that:

$$\rho^{\overline{N}} u'(\zeta(\delta/N))K(\overline{y}) < (\varepsilon/2) \tag{47}$$

Now let $T \in \mathbb{N}$ be such that $T \geq \overline{N}$. Define:

$$A(T;N) = \{\omega \in \Omega : \mathbf{y}_T < (\delta/N)\}; D(T;N) = \{\omega \in \Omega : \mathbf{y}_T \geq (\delta/N)\} \tag{48}$$

Then, we can write:

$$E\{\mathbf{p}_T \mathbf{x}_T\} = \int_{A(T;N)} \{\mathbf{p}_T \mathbf{x}_T\} P(d\omega) + \int_{D(T;N)} \{\mathbf{p}_T \mathbf{x}_T\} P(d\omega) \tag{49}$$

The first integral on the right-hand side of (49) can be evaluated as:

$$\begin{aligned}\int_{A(T;N)} \{\mathbf{p}_T\mathbf{x}_T\} P(d\omega) &\leq (\delta/N)\int_{A(T;N)} \mathbf{p}_T P(d\omega) \\ &\leq (\delta/N)\int_{A(T)} \mathbf{p}_T P(d\omega) \\ &\leq (\delta/N)(2Q'/\delta) < (\varepsilon/2) \end{aligned} \tag{50}$$

where we have used the fact that $A(T;N) \subset A(T)$ in the second line of (50) [since $N \geq 2$], and we have used (44) and (46) in the third line of (50).

The second integral on the right-hand side of (49) can be evaluated as:

$$\begin{aligned}\int_{D(T;N)} \{\mathbf{p}_T\mathbf{x}_T)\} P(d\omega) &\leq \rho^T u'(\zeta(\delta/N))K(\overline{y})\int_{D(T;N)} P(d\omega) \\ &\leq \rho^T u'(\zeta(\delta/N)))K(\overline{y}) \\ &\leq \rho^{\overline{N}} u'(\zeta(\delta/N))K(\overline{y}) < (\varepsilon/2) \end{aligned} \tag{51}$$

where in the first line of (51), we have used the fact that on $D(T;N)$, we have by (32), $\mathbf{c}_T \geq \eta\mathbf{y}_T \geq \zeta(\delta/N)$, and so $\mathbf{p}_T = \rho^T u'(\mathbf{c}_T)) \leq \rho^T u'(\zeta(\delta/N))$,while $\mathbf{x}_T \leq K(\overline{y})$. In the last line of (51), we have used $T \geq \overline{N}$, and (47). Using (50) and (51) in (49), we get (45). This establishes the transversality condition (26).

Step 3: Optimality

Finally, we show that $c(y)$ is optimal. For any feasible stochastic process of output, consumption and investment $\{\widehat{\mathbf{y}}_t, \widehat{\mathbf{c}}_t, \widehat{\mathbf{x}}_t\}$ from initial stock $\overline{y}$, the monotone convergence theorem[23] implies

$$\begin{aligned} E\left\{\sum_{t=0}^{\infty}\rho^t u(\widehat{\mathbf{c}}_t)\right\} - E\left\{\sum_{t=0}^{\infty}\rho^t u(\mathbf{c}_t))\right\} &= \lim_{T\to\infty} E\left\{\sum_{t=0}^{T}\rho^t u(\widehat{\mathbf{c}}_t) - \sum_{t=0}^{T}\rho^t u(\mathbf{c}_t))\right\} \\ &\leq \lim_{T\to\infty}\sup\,[E\{\mathbf{p}_T\mathbf{x}_T\} - E\{\mathbf{p}_T\widehat{\mathbf{x}}_T\}] \leq 0. \end{aligned}$$

where the first inequality uses (25) and the second inequality uses the transversality condition (26). Hence, $c(y)$ is optimal. This completes the proof of the lemma. ■

[23] $\sum_{t=0}^{T}\rho^t[u(K(\overline{y})) - u(\mathbf{c}_t))]$ and $\sum_{t=0}^{T}\rho^t[u(K(\overline{y})) - u(\widehat{\mathbf{c}}_t))]$ are non-negative and non-decreasing in T.

5 Non-optimal Continuous Ramsey-Euler Policy

Our main result shows that in the canonical stochastic optimal growth model analyzed in this paper, continuity of a policy function that meets the (Ramsey) Euler condition is necessary and sufficient for it to be optimal. However, continuity may not be sufficient for optimality of a Ramsey-Euler policy function in an extended version of the same model which allows for unbounded expansion of consumption.

We provide below an example of a smooth and convex optimal growth model where a continuous (and monotonic) Ramsey-Euler consumption policy function is not optimal.[24] The example satisfies all of the assumptions made in this paper with the exception of assumption (**T.2**)(i). In particular, the production technology in this example allows for unbounded expansion of consumption and output.

Example 3.

Define the utility function u to be:

$$u(c) = \frac{\sqrt{c}}{1+\sqrt{c}} \text{ for all } c \geq 0$$

Then, u satisfies (U.1). The production technology is deterministic (alternatively, the i.i.d random shocks have a degenerate distribution). The deterministic production function is linear and given by

$$f(x) = 2x$$

which satisfies (**T.1**) and (**T.2**)(ii). The discount factor is chosen to be $\rho = (1/2)$. Consider the dynamic optimization problem discussed in Section 2.3. Note that u is bounded with $u(c) \in [0, 1]$ for all $c \geq 0$; therefore, the dynamic optimization problem is well-defined and standard dynamic programming arguments ensure the existence of an optimal consumption function.

Consider the consumption function defined by:

$$c(y) = \begin{cases} (1/2)y & \text{for } 0 \leq y \leq 2 \\ 1 & \text{for } y > 2 \end{cases}$$

Observe that $c(y)$ is interior and continuous; further, $c(y)$ and $y - c(y)$ are non-decreasing in y. We will now show that the consumption function $c(\cdot)$ satisfies the

[24] The example settles a query raised by an anonymous referee.

Ramsey-Euler condition (RE). For $0 < y \leq 2$, we have $c(y) = (1/2)y$, and $f(y - c(y)) = 2(y - (1/2)y) = y$, so that $c(f(y - c(y))) = (1/2)y = c(y)$. Thus

$$\rho u'(c\{f(y - c(y))\})f'(y - c(y)) = \frac{1}{2}u'((1/2)y)2 = u'(c(y))$$

verifying (RE) for $y \in (0, 2]$. For $y > 2$, we have $2(y - 1) = 2y - 2 > 2$, and so $c\{f(y - c(y))\} = c\{2(y - 1)\} = 1$. Thus,

$$\begin{aligned} \rho u'(c\{f(y - c(y))\})f'(y - c(y)) &= (1/2)u'(c\{f(y - c(y))\})2 \\ &= u'(1) = u'(c(y)) \end{aligned}$$

verifying (RE) for $y > 2$. Finally, we show that $c(y)$ is not an optimal consumption function. To see this, consider the consumption function $\gamma(\cdot)$ defined by:

$$\gamma(y) = (1/2)y \ \text{ for all } y \geq 0$$

Starting from $y = 4$, the consumption function $\gamma(\cdot)$ generates a path $(\widetilde{y}_t, \widetilde{c}_t, \widetilde{x}_t)$ where consumption $\widetilde{c}_t = 2$ for all $t \geq 0$. On the other hand, the path (y_t, c_t, x_t) starting from $y = 4$, generated by the consumption function $c(\cdot)$, has $y_t \geq 4$ for all $t \geq 0$ and so $c_t = 1$ for all $t \geq 0$, so that the discounted sum of utilities along the path (y_t, c_t, x_t) is strictly smaller than along the path $(\widetilde{y}_t, \widetilde{c}_t, \widetilde{x}_t)$. This shows that $c(\cdot)$ is *not* an optimal consumption function.[25] This concludes the example. ■

Example 3 indicates that in order to extend our result to a larger class of dynamic optimization problems, one may need to expand the set of conditions that the policy function needs to satisfy.

6 Conclusion

In the standard one sector model of stochastic optimal growth, we have shown that to establish that an interior Ramsey-Euler policy function is optimal, it is not necessary to verify the transversality condition; such a policy function is optimal as long it is continuous since this ensures (as we have demonstrated in the proof of Theorem 1)

[25] One can also check directly that the transversality condition (TC) is violated by the path (y_t, c_t, x_t) since $y_t = [2 + 2^{t+1}]$, $p_t = [u'(1)/2^t]$ for all $t \geq 0$.

that the transversality condition is satisfied. Further, continuity is ensured if both consumption and investment are non-decreasing in current output and so the latter are also sufficient for an interior Ramsey-Euler policy to be optimal. This allows us to state a new set of necessary and sufficient conditions for dynamic optimality that should be useful in various applications of the model.

The broad implication of our main result is that in certain stochastic dynamic economic models it may be possible to verify optimality of a policy function that satisfies the Euler equation without verifying that the stochastic process generated by the policy function satisfies a transversality condition as long as certain global properties of the policy function hold. However, the precise properties of the policy function that one needs to verify in order to avoid checking the transversality condition are likely to depend on the structure of the model. Future research should focus on characterizing alternative conditions for optimality in a larger class of dynamic optimization models than the one considered in this paper.[26]

7 Appendix

Proof of Lemma 1: For any $y_1, y_2 \in \mathbb{R}_+$,

$$c(y_1) - c(y_2) = (y_1 - y_2) - (x(y_1) - x(y_2))$$

As $c(y)$ and $x(y)$ are non-decreasing on $\mathbb{R}_+$,

$$0 \leq c(y_1) - c(y_2) \leq (y_1 - y_2), \text{ if } y_1 \geq y_2$$

$$0 \geq c(y_1) - c(y_2) \geq (y_1 - y_2), \text{ if } y_1 \leq y_2$$

Thus,

$$|c(y_1) - c(y_2)| \leq |y_1 - y_2|$$

which implies that $c(y)$ is continuous.[27] ■

Some Details of Example 2: First, we verify that $c(y)$ is an interior consump-

[26] In particular, extension of our results to stochastic growth models where production technologies allow for unbounded expansion of output (for instance, Levhari and Srinivasan 1969, de Hek 1999, de Hek and Roy 2001), "unbounded shocks" (see, Stachurski 2002, Nishimura and Stachurski 2005, Kamihigashi 2007) and irreversible investment (see, Olson 1989) will be very useful.

[27] This simplified proof was suggested by an anonymous referee.

tion function. On $A = (0, 4]$, we have $c(y) = (1 - (\rho/2))y$, so $0 < c(y) < y$, since $(\rho/2) \in (0, 1)$ and $y > 0$. On $B = (4, \infty)$, $c(y) = (1 - \rho)(y - 4) > 0$ since $y > 4$ and $\rho \in (0, 1)$; further $c(y) = (1 - \rho)y - 4(1 - \rho) < (1 - \rho)y < y$, since $\rho \in (0, 1)$, and $y > 0$.

Next, we show that if the initial stock $y \in A$, then the sequence $\{y_t, c_t, x_t\}$ generated by c has the property that $y_t \in A$ for all $t \geq 0$. Similarly, if the initial stock $y \in B$, then the sequence $\{y_t, c_t, x_t\}$ generated by c has the property that $y_t \in B$ for all $t \geq 0$.

The first observation can be seen as follows. For $y \in A$, the investment function is given by:

$$x(y) = y - c(y) = (\rho/2)y \in A \text{ for all } y \in A$$

Consequently, we have:

$$f(x(y)) = 2[x(y)]^{1/2} = 2(\rho/2)^{(1/2)}y^{1/2} \in A \text{ for all } y \in A$$

For the second observation, note that for $y \in B$, the investment function is given by:

$$x(y) = y - c(y) = y - (1 - \rho)(y - 4) = \rho y + (1 - \rho)4 \in B \text{ for all } y \in B$$

Consequently, we have:

$$f(x(y)) = 2 + (1/2)[\rho y + (1 - \rho)4] = 2 + 2(1 - \rho) + (1/2)\rho y \in B \text{ for all } y \in B$$

Finally, we verify explicitly that the sequence $\{y_t, c_t, x_t\}$ from $y = 5$ does not satisfy the transversality condition. In particular, for every $t \geq 0$,

$$\begin{aligned} c_t &= (1 - \rho)(y_t - 4), x_t = \rho y_t + 4(1 - \rho), y_{t+1} = \frac{\rho}{2} y_t + 4 - 2\rho \\ c_{t+1} &= \frac{1}{2}\rho(1 - \rho)(y_t - 4), x_{t+1} = \frac{\rho^2}{2} y_t + 4 - 2\rho^2 \end{aligned}$$

From (5) and $y_{t+1} = \frac{\rho}{2} y_t + 4 - 2\rho$ we have that $y_t \to 4$ as $t \to \infty$. In consequence, as

$t \to \infty$,

$$\frac{\rho^{t+1}u'(c_{t+1})x_{t+1}}{\rho^t u'(c_t)x_t} = \rho \frac{\left[\frac{\rho^2}{2}y_t + 4 - 2\rho^2\right]}{[\rho y_t + 4(1-\rho)]} \frac{[(1-\rho)(y_t - 4)]}{\left[\frac{1}{2}\rho(1-\rho)(y_t-4)\right]}$$
$$= \frac{\rho^2 y_t + 4(2-\rho^2)}{\rho y_t + 4(1-\rho)} \to 2$$

so that $\rho^t u'(c_t)x_t$ diverges to $+\infty$ as $t \to \infty$. ■

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