Lectures on Trading with Information

Competitive Noisy Rational Expectations Equilibrium (Grossman and Stiglitz AER (1980))

1 Assumptions

(A1) Two Assets: Trading in the asset market involves a risky asset and a riskless asset which serves as the numeriare. Normalize the riskless rate of return to one. The risky asset generates a stochastic return $\tilde{v}$ which is normally distributed with mean $\bar{v}$ and variance σ_v^2. Without loss of generality, set $\sigma_v^2 = 1$.

(A2) Homogenous Investors: There is a continuum of atomistic investors with a total mass one. Each investor is assumed to be endowed with an initial wealth w_0. Investors have identical CARA preferences represented by the utility function

$$U(W) = -\exp(-aW) \tag{1}$$

where W is the final wealth and $a > 0$ is the coefficient of absolute risk aversion common to all investors. Furthermore, a priori all investors possess the same information on the return of the risky asset, summarized by the distribution of the return.

(A3) Competitive Rational Expectations Equilibrium: The market for the risky asset clears in per capita sense and each investor maximizes the expected utility from final wealth conditional on all the information available, *including the equilibrium asset price*. In choosing their portfolios, investors take the price distribution and price realizations as given.

(A3) Noisy Supply: The per capita supply $\tilde{x}^s$ of the risky asset is also normally distributed with zero mean and variance σ_x^2 and is independent of any information that traders may have.

(A4) Information: By paying a fixed cost $c > 0$, investors can observe an information signal on the return of the risky asset where

$$\tilde{v} = \tilde{s} + \tilde{\varepsilon} \tag{2}$$

Furthermore, $\tilde{s}$ and $\tilde{\varepsilon}$ have a multivariate normal distribution with

$$E\tilde{\varepsilon} = 0,\ E\tilde{s}\tilde{\varepsilon} = 0 \text{ and } Var\ \tilde{\varepsilon} \equiv h^{-1} > 0 \tag{3}$$

Since $\tilde{s}$ and $\tilde{\varepsilon}$ are uncorrelated, (3) implies that the posterior mean is

$$E\left[\tilde{v}|s\right] \equiv \mu(s) = s \tag{4}$$

and the posterior variance conditional on signal is

$$Var\left[\tilde{v}|s\right] = h^{-1} \tag{5}$$

2 Asset Market Equilibrium

2.1 Portfolio Problem of Investors

2.1.1 Informed Investors:

Upon observing the information signal s, an informed investor chooses her position in the risky asset to maximize her final portfolio wealth. One can write the *conditional* distribution of her portfolio wealth as

$$\tilde{W}_I(s,p) = rw_0 + D_I(\tilde{v} - rp) - c \tag{6}$$

where D_I is the position in the risky asset. The portfolio problem is

$$\underset{D_I}{Max} E\left[-exp(-a\tilde{W}_I)|s,p\right]. \tag{7}$$

Due to normality and CARA preferences, this can be stated as a mean variance problem

$$\underset{D_I}{Max} E\left[\tilde{W}_I|s,p\right] - \frac{a}{2} Var\left[\tilde{W}_I|s,p\right] \tag{8}$$

The informed investor's demand for the risky asset is given by

$$D_I^*(s,p) = h(\frac{s-p}{a}). \tag{9}$$

2.1.2 Uninformed Investors

Suppose that the uninformed investors conjecture that the equilibrium asset price is of the form

$$p^*_\lambda(s, x^s) = \beta s - \gamma x^s \tag{10}$$

where β and γ are equilibrium coefficients. An uninformed investor chooses her demand D_u for the risky asset to maximize the expected utility from her portfolio

$$E[-\exp(-a\tilde{W}_U)|p^*_\lambda] = E[-\exp(-a(D_u(\tilde{v} - rp) + rw_0)|p^*_\lambda]. \tag{11}$$

In doing so, she takes the price realization as given, but uses the equilibrium asset price as an informative signal through her conjecture. In order to see how an uninformed investor forms posteriors on $\tilde{v}$ using her conjecture p^*_λ, define a random variable $\tilde{\theta}$ using the conjecture as;

$$\tilde{\theta}(s, x_s) \equiv \frac{p^*_\lambda(s, x_s)}{\beta} = \tilde{s} - \frac{\gamma}{\beta}\tilde{x}_s. \tag{12}$$

Conditioning on $\tilde{\theta}$ is statistically equivalent to conditioning on the conjecture $p^*_\lambda(s, x_s)$. To see how $\tilde{\theta}$ conveys information about $\tilde{v}$, substitute $\tilde{v} = \tilde{s} + \tilde{\epsilon}$ in the above expression;

$$\tilde{\theta} = \tilde{v} - (\tilde{\epsilon} + \frac{\gamma}{\beta}\tilde{x}_s) \Rightarrow \tilde{v} = \tilde{\theta} + (\tilde{\epsilon} + \frac{\gamma}{\beta}\tilde{x}_s). \tag{13}$$

Without any supply uncertainty, conditioning on $\tilde{\theta}$ is informationally equivalent to conditioning on the signal and therefore all the information is revealed through the equilibrium asset price. With supply uncertainty, the asset price becomes a noisy indicator of the signal. The optimal uninformed demand D_u that maximizes the uninformed portfolio wealth is given by;

$$D^*_u = \frac{E\left[\tilde{v}|p^*_\lambda\right] - p}{aVar\left[\tilde{v}|p^*_\lambda\right]}. \tag{14}$$

The extent of information revealed by price is given by the uninformed posterior variance $Var\left[\tilde{v}|p^*_\lambda\right]$. In an equilibrium where the equilibrium asset price has the linear form as above, it follows that the uninformed posterior variance (inverse of precision) is;

$$\frac{1}{\hat{h}^u} \equiv Var\left[\tilde{v}|p^*_\lambda\right] = 1 - \frac{(1 - \hat{h}^{-1})^2}{(1 - \hat{h}^{-1}) + (\frac{\gamma}{\beta})^2\sigma^2_x}. \tag{15}$$

The ratio of equilibrium coefficients β and γ is of particular importance for the extent of information revealed from the equilibrium asset price. If γ/β is small in absolute terms, then the supply uncertainty becomes less important relative to the information signal in determining price movements and uninformed traders are able to extract more information from the equilibrium asset price.

Proposition 1 *(i) There is a rational expectations equilibrium where the equilibrium asset price is linear in signal realization and per capita supply;* $p(s, x^s) = \beta s - \gamma x^s$. *The ratio of equilibrium coefficients is given by;*

$$\frac{\gamma}{\beta} = \frac{a}{\lambda h} \tag{16}$$

(ii) The informativeness of the equilibrium asset price is increasing in the precision h *of information and the sales volume* λ *and it is decreasing in the risk aversion coefficient* a

Lecture 2: Strategic Informed Trader
(Kyle 1985, Econometrica)

3 Assumptions

Informed Trader: A single risk neutral informed trader receives private information about the ex post liquidation value, v, of a risky asset.

This liquidation value is assumed to be normally distributed with mean p_0 and variance Σ.

$$\tilde{v} \sim N(p_0, \Sigma) \tag{17}$$

Noise Traders: Other than the informed trader, it is assumed that there are noise traders who do not act strategically, but rather trade a random amount u, which is normally distributed with mean 0 and variance ϕ.

$$\tilde{u} \sim N(0, \phi) \tag{18}$$

Market Maker: There is a market maker who sets prices. The market maker does not observe the individual orders by traders, but only observes the aggregate order flow X. In that sense this a batch-clearing model.

Trading Protocol: The trading protocol involves a two-step process. In the first step, v (the asset's true value is realized and observed only by the informed trader. The informed trader chooses his trade quantity x. In the second stage, the market maker observes an aggregate order flow

$$X = x + u \tag{19}$$

where u is the realized noise trade. Again, the market maker does not observe x and u separately. The market maker sets the asset price p equal to

$$p = E\left[\tilde{v}|X\right].$$

The informed trader chooses a trading strategy to maximize her expected profits given by

$$\pi = E\left[x(v - \tilde{p})|v\right]$$

4 Equilibrium

Kyle shows that there is a unique linear equilibrium of this trading game where the informed trader's equilibrium trading strategy is linear in her private information. This is given by

$$x(v) = \beta(v - p_0)$$

and market maker's equilibrium pricing rule is linear in the total order flow she receives and this is given by

$$p(X) = p_0 + \lambda X$$

where

$$\begin{aligned} \beta &= \left(\frac{\phi}{\Sigma}\right)^{\frac{1}{2}} \\ \lambda &= \frac{1}{2}\left(\frac{\Sigma}{\phi}\right)^{\frac{1}{2}} \end{aligned}$$

Proof:

Market Maker: Given the trading strategy of the informed trader

$$x(v) = \beta(v - p_0) \tag{20}$$

the market maker sets

$$p = E\left[\tilde{v}|X\right] \tag{21}$$

Now, to illustrate the market maker's signal extraction problem, note that

$$X = \beta v + u - \beta p_0 \tag{22}$$

and thus we can define

$$Z \equiv \frac{X + \beta p_0}{\beta} = v + \frac{1}{\beta}u \tag{23}$$

Conditioning on Z is statistically equivalent to conditioning on X. Therefore, we have

$$\begin{aligned} E\left[\tilde{v}|X\right] &= E\left[\tilde{v}|Z\right] \\ E\left[\tilde{v}|Z\right] &= \frac{\frac{p_0}{\Sigma} + Z(\frac{\beta^2}{\phi})}{\frac{1}{\Sigma} + \frac{\beta^2}{\phi}} = \frac{p_0\phi + \beta^2\Sigma Z}{\phi + \beta^2\Sigma} \end{aligned} \tag{24}$$

Now, substitute back $Z \equiv \frac{X+\beta p_0}{\beta}$ and get

$$E[\tilde{v}|Z] = E[\tilde{v}|X] = p_0 + \left(\frac{\beta\Sigma}{\phi + \beta^2\Sigma}\right) X \tag{25}$$

Therefore,

$$p = p_0 + \left(\frac{\beta\Sigma}{\phi + \beta^2\Sigma}\right) X$$

which implies

$$\lambda = \frac{\beta\Sigma}{\phi + \beta^2\Sigma} \tag{26}$$

Informed Trader: Given the pricing strategy of the market maker,

$$p(X) = p_0 + \lambda X \tag{27}$$

informed trader chooses x to maximize expected profits π :

$$\begin{aligned} \pi &= E[x(v - \tilde{p})|v] = xv - xE(\tilde{p}) \\ &= xv - x(p_0 + \lambda x) \\ &= xv - xp_0 - \lambda x^2 \end{aligned} \tag{28}$$

Well, that's easy. Optimal trading quantity x is given by

$$\begin{aligned} v - p_0 - 2\lambda x &= 0 \\ x &= \frac{v - p_0}{2\lambda} \end{aligned} \tag{29}$$

which implies that (since we are looking for $x = \beta(v - p_0)$)

$$\beta = \frac{1}{2\lambda} \tag{30}$$

Solving this together with

$$\lambda = \frac{\beta\Sigma}{\phi + \beta^2\Sigma}$$

yields the equilibrium coefficients β and λ.

5 Application: Central Bank Intervention (Vitale (1999)-Journal of International Economics)

Vitale (1999) adopts a Kyle-type microstructure framework in the foreign exchange market and analyzes whether the central bank should keep its target secret from market participants.

The basic ingredients of his model are as follows:

Market Participants

(i) Market Maker: There is a risk neutral dealer (the 'market maker') who trades the foreign currency with a central banks and a group of liquidity traders. Prior to trades, the fundamental value of the exchange rate, f, is known only to the central bank.

For the market maker, f is a normal random variable with mean s_0 and variance Σ. At the time of the trading, the market maker calls an auction for the currency and observes a total order flow X. Competition between market makers and the risk neutrality assumption imply that the equilibrium exchange rate s_1 set by the market maker is given by the following zero profit condition:

$$s_1 = E\left[f|X\right]. \tag{31}$$

(ii) Central Bank: Central bank knows the exchange rate fundamental value f perfectly. The bank submits a market order x to minimize the expected value of her loss function:

$$c \equiv (s_1 - f)x + q(s_1 - \bar{s})^2 \tag{32}$$

with s_1 set according to the market maker's pricing rule in (1) above. The first part of the loss function, $(s_1 - f)x$ reflects bank's monetary losses or gains, whereas the second part, $q(s_1 - \bar{s})^2$ describes the targeting agenda: $\bar{s}$ is the bank's exchange rate target and $q \geq 0$ describes the central bank's commitment to that target.

(iii) Liquidity Traders: The market order of the liquidity traders, ε, is a normal random variable with mean zero and variance σ_ε^2 and it is independent from the fundamental value f.

Accordingly, the total order flow X that the market maker receives is

$$X = x + \varepsilon$$

Information Structure

Depending on the Central Bank's disclosure policy, the other market participants may or may not know the target $\bar{s}$, but its commitment to the target, q, is common knowledge. Without disclosure, the prior distribution of $\bar{s}$ is normal with mean $\hat{s}$ and variance σ_s^2 and it is independent from the fundamental value f. Within this setting, Vitale analyzes two possibilities as regard to who knows the target $\bar{s}$ at the time of the trading.

(i) The target is common knowledge.

Proposition 2 *(Vitale (1999)) When the central bank's target $\bar{s}$ is publicly disclosed, intervention has no effect and the central bank cannot target the exchange rate.*

When the target is common knowledge, the unique linear equilibrium of the trading game is

$$x = \beta(f - s_0) + 2q(\bar{s} - s_0)$$

and

$$s_1 = s_0 + \lambda[\beta(f - s_0) + \varepsilon]$$

where

$$\beta = \frac{1}{2\lambda(1 + \lambda q)}$$

and λ is the unique positive root of the following equation:

$$4\lambda^2(1 + \lambda q)^2\sigma_\varepsilon^2 = (1 + 2\lambda q)\Sigma$$

(ii) The target is secret.

Proposition 3 *(Vitale (1999)) When the central bank's target $\bar{s}$ is secret, the intervention has an effect and the central bank can target the exchange rate.*

When the target is secret, the unique linear equilibrium of the trading game is,

$$x = \beta(f - s_0) + 2q(\bar{s} - s_0) + \theta(\bar{s} - \hat{s})$$

and

$$s_1 = s_0 + \lambda[\beta(f - s_0) + \theta(\bar{s} - \hat{s}) + \varepsilon]$$

where

$$\beta = \frac{1}{2\lambda(1 + \lambda q)} \text{ and } \theta = \frac{q}{1 + \lambda q}$$

and λ is the unique positive root of the following equation:

$$4\lambda^2[(1 + \lambda q)^2\sigma_\varepsilon^2 + \sigma_s^2] = (1 + 2\lambda q)\Sigma$$

Risk Aversion, Market Liquidity, and Price Efficiency (Subrahmanyam 1991, Review of Financial Studies)

6 Assumptions

Subrahmanyam (1991) introduces risk aversion to informed speculator in the original Kyle model and derives a relationship between the number of speculators and market liquidity.

The model is the same as Kyle (1985), except the following three features:

(i) There are k informed traders (speculators).

(ii) The information of the speculator's is not perfect. Rather than observing the liquidation value of the asset perfectly, they observe a noisy signal.

Let F denote the liquidation value of the asset. Subrahmanyam (1991) assumes that

$$F = \bar{F} + \tilde{\delta} \tag{33}$$

where $\bar{F}$ is known by everyone and informed traders observe a signal s

$$\tilde{s} = \tilde{\delta} + \tilde{u}. \tag{34}$$

The random noise trade is denoted by z. All random variables are normally distributed with mean zero. Let

$$Var(\tilde{u}) = \phi,\ Var(\tilde{z}) = \sigma_z^2 \text{ and } Var(\tilde{\delta}) = 1 \tag{35}$$

(iii) Finally, the informed traders have CARA preferences, with common coefficient of risk aversion A.

7 Equilibrium

We will again characterize an equilibrium where trading sttegies and market maker's pricing rule is linear. The market maker sets the price P as

$$P = E\left[F|\omega\right] \tag{36}$$

where ω is the aggregate order she receives. As before, suppose the market maker employs a linear pricing rule, which is consistent with the above zero profit condition, as we showed in Lecture 2;

$$P = \bar{F} + \lambda\omega \tag{37}$$

Denote a particular informed trader's order by x. Expected utility from trading on the signal $\delta + u$ is

$$-E[\exp(-AX(F - P))|\delta + u]. \tag{38}$$

Let this informed trader conjecture that each of the other informed traders submit an order $\beta(\delta + u)$. Conditional on the signal s, the trader's profits from trading on information are normally distributed. By CARA preferences, this implies that we can write her objective function in the mean-variance form

$$\begin{aligned} &E[x(\delta - \lambda(x + (k-1)\beta(\delta + u) + z|\delta + u] \\ &-(A/2)Var[x(\delta - \lambda(x + (k-1)\beta(\delta + u) + z|\delta + u] \end{aligned} \tag{39}$$

Differentiating this with rerspect to x, we arrive

$$x = \frac{\delta + u}{(1+\phi)(2\lambda + A\lambda^2\sigma_z^2) + A\phi} - \frac{(k-1)\beta\lambda(\delta + u)}{(1+\phi)(2\lambda + A\lambda^2\sigma_z^2) + A\phi} \tag{40}$$

where the second term on the RHS is the amount by which informed traders scale down their orders because of the presence of other informed traders.

We can now obtain the unique linear Nash equilibrium by setting $x = \beta(\delta + u)$ in the above equation and solving for β :

$$\beta = \frac{1}{(1+\phi)\lambda(k+1) + A[\phi + \lambda^2\sigma_z^2(1+\phi)]} \tag{41}$$

So, we see that the effect of risk aversion and noisy information is to make the informed trade less aggressively on her information.

We can also solve for the liquidity coefficient λ. The equilibrium λ solves the following equation

$$\lambda[k^2(1+\phi) + \alpha^2\sigma_z^2] = k\alpha \tag{42}$$

where

$$\alpha = \frac{1}{\beta}. \tag{43}$$

If the informed were risk neutral, λ would be given by

$$\lambda = \frac{1}{k+1}\sqrt{\frac{k}{\sigma_z^2(1+\phi)}} \tag{44}$$

7.1 Comparative Statics

(i) How does market liquidity depend on the number of informed traders?

When informed traders are risk neutral, λ (market depth-inverse of liquidity) is decreasing in k. However, with risk aversion, market liquidity is non-monotonic in k, their risk aversion A and the precision of their information.

(ii) How does price efficiency depend on the number of informed traders andf their risk aversion?

Price efficiency is the extent that prices reveal private information. Subrahmanyam defines informational efficiency (informativeness of price) as the posterior precision of δ conditional on the price (or the order flow, since they are linearly related) as

$$Q = [Var(\delta|\omega)]^{-1} \tag{45}$$

It can be shown that

$$Q = 1 + \frac{1}{\phi + (\frac{\sigma_z}{k\beta})^2} \tag{46}$$

Note that the above expression is not closed form, since β is endogenous. However, it is straightforward to see that as the trading intensity of the informed traders β increases, more private information is relealed from the price. Therefore, price efficiency is decreasing in risk aversion A of the informed traders Furthermore, an increase in the variance of the liquidity trade σ_z^2 also decreases price efficiency.