Question  Consider the following Prisoners’ Dilemma

<table>
<thead>
<tr>
<th></th>
<th>D</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>1,1</td>
<td>6,0</td>
</tr>
<tr>
<td>C</td>
<td>0,6</td>
<td>4,4</td>
</tr>
</tbody>
</table>

Suppose the above stage game played infinitely.
Each player has a discount factor $\delta \in (0, 1)$.
For what values of $\delta \in (0, 1)$, if any, does the following strategy constitute subgame
perfect equilibria?

Tit-For-Tat: Play C in the first period. Then, do whatever your opponent did in the
last period.

Answer: The behavior only depends on the last period’s outcome. Therefore, we can
group all the histories into those with the last period’s outcome being (C,C), (C,D), (D,C)
and (D,D). Now we need to check the optimality of the tit-for-that strategy after all such
histories using the one-shot-deviation property.

Optimality after histories that end with (C,C). Following the equilibrium strategy
results in the following continuation path and period payoffs

$$(C, C), (C, C), (C, C), (C, C).....$$

$$(C, C), (C, C), (C, C), (C, C).....$$

Hence the discounted payoff from following the equilibrium strategy is

$$4 + 4\delta + 4\delta^2 + 4\delta^3 + 4\delta^4 + .... = \frac{4}{1-\delta}$$

Deviating and playing D results in

$$(D, C), (C, D), (D, C), (C, D).....$$

$$(D, C), (C, D), (D, C), (C, D).....$$

which gives a discounted payoff

$$6 + 6\delta^2 + 6\delta^4 + 6\delta^6 + 6\delta^8 + .... = \frac{6}{1-\delta^2}$$

Therefore, tit-for-tat is optimal if

$$\frac{4}{1-\delta} \geq \frac{6}{1-\delta^2} \Rightarrow \delta \geq \frac{1}{2}. \quad (1)$$
Optimality after histories that end with (C,D). Following the equilibrium strategy results in the following continuation path and period payoffs

\[(D, C), (C, D), (D, C), (C, D)......\]

\[6, 0, 6, 0, 6, 0.....\]

Hence the discounted payoff from following the equilibrium strategy is

\[6 + 6\delta^2 + 6\delta^4 + 6\delta^6 + 6\delta^8 + .... = \frac{6}{1 - \delta^2}\]

Deviating and playing C results in

\[(C, C), (C, C), (C, C), (C, C)......\]

\[4, 4, 4, 4, 4......\]

which gives a discounted payoff

\[4 + 4\delta + 4\delta^2 + 4\delta^3 + 4\delta^4 + .... = \frac{4}{1 - \delta}\]

Therefore, tit-for-tat is optimal if

\[
\frac{6}{1 - \delta^2} \geq \frac{4}{1 - \delta} \Rightarrow \delta \leq \frac{1}{2}.
\]  \hspace{1cm} (2)

Optimality after histories that end with (D,C). Following the equilibrium strategy results in the following continuation path and period payoffs

\[(C, D), (D, C), (C, D)......\]

\[0, 6, 0, 6, 0.....\]

Hence the discounted payoff from following the equilibrium strategy is

\[6\delta + 6\delta^3 + 6\delta^5 + 6\delta^7 + .... = \frac{6\delta}{1 - \delta^2}\]

Deviating and playing D results in

\[(D, D), (D, D), (D, D), (D, D)......\]

\[1, 1, 1, 1, 1......\]
which gives a discounted payoff

\[ 1 + \delta + \delta^2 + \delta^3 + \delta^4 + \ldots = \frac{1}{1 - \delta} \]

Therefore, tit-for-tat is optimal if

\[ \frac{6\delta}{1 - \delta^2} \geq \frac{1}{1 - \delta} \Rightarrow \delta \geq \frac{1}{5} \]  \((3)\)

Optimality after histories that end with (D,D). Following the equilibrium strategy results in the following continuation path and period payoffs

\((D, D), (D, D), (D, D), (D, D) \ldots \)

1, 1, 1, 1, 1......

Hence the discounted payoff from following the equilibrium strategy is

\[ 1 + \delta + \delta^2 + \delta^3 + \delta^4 + \ldots = \frac{1}{1 - \delta} \]

Deviating and playing C results in

\((C, D), (D, C), (C, D) \ldots \)

0, 6, 0, 6, 0....

which gives a discounted payoff

\[ 6\delta + 6\delta^3 + 6\delta^5 + 6\delta^7 + \ldots = \frac{6\delta}{1 - \delta^2} \]

Therefore, tit-for-tat is optimal if

\[ \frac{1}{1 - \delta} \geq \frac{6\delta}{1 - \delta^2} \Rightarrow \delta \leq \frac{1}{5} \]  \((4)\)

Putting the conditions in (1), (2), (3) and (4) together, we conclude that there is no \( \delta \) such that tit-for-tat strategy profile is SPE.