Two Performance Outcomes

- Output is denoted by $q \in \{0, 1\}$. Costly effort by the agent makes high output more likely.

$$\Pr(q = 1 \mid a) = p(a) \text{ with } p' > 0 \text{ and } p'' < 0.$$

- Principal's utility is $V(q - w)$ and agent's utility is $u(w) - \psi(a)$. Assume $\psi(a) = a$.

First Best

- **The principal's problem is**

$$\underset{(a,w_0,w_1)}{Max}\; p(a)V(1-w_1)+(1-p(a))V(-w_0) \qquad \text{subject to}$$

$$p(a)u(w_1)+(1-p(a))u(w_0)-a \geq \bar{u} \qquad (1)$$

- **First order conditions**

$$p'(a)[V(1-w_1) - V(-w_0)] + \lambda p'(a)[u(w_1) - u(w_0)] - \lambda = 0$$

$$\lambda u'(w_1) = V'(1-w_1)$$

$$\lambda u'(w_0) = V'(-w_0)$$

$$\Longrightarrow \frac{V'(1-w_1)}{u'(w_1)} = \frac{V'(-w_0)}{u'(w_0)} = \lambda \text{ (Borsch's Rule)}$$

Risk Neutral Principal $\mathbf{V(x) = x}$

- We have

$$\frac{1}{u'(w_1)} = \frac{1}{u'(w_0)} = \lambda \Longrightarrow w_0 = w_1 = w^*$$

$$p(a)u(w_1) + (1 - p(a))u(w_0) - a \geq \bar{u} \text{ (IRC)}$$

$$\Longrightarrow u(w^*) = a$$

$$p'(a)[V(1 - w_1) - V(-w_0)] + \lambda p'(a)[u(w_1) - u(w_0)] - \lambda = 0$$

$$p'(a)[1 - w_1 + w_0)] - \lambda = 0 \Longrightarrow p'(a^{FB}) = \frac{1}{u'(w^*)}$$

- **Second Best: The principal's problem is**

$$\underset{(a,w_0,w_1)}{Max}\, p(a)V(1-w_1)+(1-p(a))V(-w_0) \qquad \text{subject to}$$

$$p(a)u(w_1)+(1-p(a))u(w_0)-a \geq \bar{u} \tag{2}$$

$$a \in \arg\max_{\hat{a}} p(\hat{a})u(w_1)+(1-p(\hat{a}))u(w_0)-\hat{a}$$

- **First Order Condition of the Agent's Effort Problem**

$$p'(a)[u(w_1) - u(w_0)] = 1$$

- Case 1: Risk Neutral Agent and No Limited Liability

$$p'(a)(w_1 - w_0) = 1$$

By setting $w_1 - w_0 = 1$ (hence $w_0 < 0$), First best can be implemented! (up-front sale of output to the agent at price $-w_0$

- Case 2: Risk Neutral Agent and Limited Liability. In this case principal sets $w_0 = 0$. Hence

$$p'(a) = \frac{1}{w_1}$$

The Principal now solves

$$\underset{(a,w_1)}{Max}\ p(a)(1 - w_1) \qquad \text{subject to } p'(a) = \frac{1}{w_1}$$

which yields

$$p'(a) = 1 - \frac{p(a)p''(a)}{(p'(a))^2} \Rightarrow a^{*SB} < a^{*FB}$$

(Since $w_0 = 0$, agent cant be pushed down to reservation and receives a surplus which is invreasing in effort induced. Hence second best involves underprovision of effort.

Technology and Preferences:

- An agent (the manager) runs a firm owned by a principal (the shareholder). The principal is risk neutral and maximizes the final firm value net of the manager's compensation.
- The manager has exponential preferences with a constant absolute risk aversion coefficient $a > 0$.

$$u(w) = -\exp(-aw)$$

- The final value of the firm $\tilde{X} = e + \tilde{\varepsilon}$ where e is the costly and unobservable effort expended by the manager and $\tilde{\varepsilon} \sim N(0, \Sigma)$
- Cost of effort is given by $c(e) = ke^2/2$ with $k > 0$.

Normal-CARA Model with Linear Contracts

Linear Compensation

- The manager's compensation contract is described by a pair (F, s) where F is a fixed payment and s is the manager's share of the final firm value. Accordingly, the manager's compensation is given by $F + s\tilde{X}$.
- We refer to s as the pay-performance sensitivity of the manager's compensation scheme.

Normal-CARA Model with Linear Contracts

Contract Problem

- The manager's final wealth is given by

$$\tilde{W}_m(e) = F + s\tilde{X} - c(e).$$

- With the normality assumption on $\tilde{\varepsilon}$ and CARA preferences, the manager's expected utility can be written in the mean-variance form. The formulation of the contract problem is as follows:

$$\underset{(F,s)}{Max}\ (1-s)E\left[\tilde{X}\right] - F \qquad \text{subject to}$$

$$E\left[\tilde{W}_m(e^*)\right] - (a/2)Var\left[\tilde{W}_m(e^*)\right] \geq 0 \qquad (3)$$

$$e^* \in \arg\max E\left[\tilde{W}_m(e)\right] - (a/2)Var\left[\tilde{W}_m(e)\right] \qquad (4)$$

Normal-CARA Model with Linear Contracts

Effort Choice

$$e^* \in \arg\max E\left[\tilde{W}_m(e)\right] - (a/2)Var\left[\tilde{W}_m(e)\right]$$

$$E\left[\tilde{W}_m(e)\right] = sE\left[X(e)\right] + F - c(e) \Rightarrow \left[\tilde{W}_m(e)\right] = se - c(e) + F$$

$$Var\left[\tilde{W}_m(e)\right] = s^2 Var\left[X(e)\right] = s^2\Sigma$$

$$e^* \in \arg\max se - c(e) - (a/2)s^2\Sigma \Rightarrow e^*(s) = \frac{s}{k}$$

Reduced Problem

- We have

$$\underset{(F,s)}{Max}\ (1-s)E\left[\tilde{X}\right]-F \qquad \text{subject to}$$

$$E\left[\tilde{W}_m(e^*)\right]-(a/2)Var\left[\tilde{W}_m(e^*)\right]\geq 0 \tag{5}$$

which can now be written as

$$\underset{(F,s)}{Max}\ (1-s)e^*(s)-F \qquad \text{subject to}$$

$$se^*(s)+F-c(e^*(s))-(a/2)s^2\Sigma\geq 0$$

Optimal Pay-Performance Sensitivity

- The optimal pay-performance sensitivity is then given by

$$s^* = \frac{1}{1 + ak\Sigma} \text{ and } e^* = \frac{1}{k(1 + ak\Sigma)} \tag{6}$$

- Risk Sharing vs Incentives Trade-off!

Risky Debt As Optimal Contract-Innes (1990)

Basic Idea

- When the costly and hidden effort by the borrower or entrepreneur (EN) raises the return on investment, then the most incentive efficient form of outside financing of the EN's project under limited liability is a debt contract.
- A debt contract of the form

$$\begin{aligned} r(q) &= D \text{ for } q \geq D \\ r(q) &= q \text{ for } q < D \end{aligned}$$

where D is set such that expected repayment is equal to the funds I borrowed.

Innes' (1990) Result

- The above debt contract provides the best incentives for effort provision by extracting as mucg as possible from the EN under low performance states and by giving her full marginal return from effort provision in high performance states.

Basic Set-up

- A risk neutral EN can raise the revenues q from an investment by incrasing effort a
- Conditional density $f(q \mid a)$ and the conditional cumulative distribution $F(q \mid a)$
- EN's utility is separable in income and effort

$$v(w, a) = w - \varphi(a) \text{ with } \varphi' > 0 \text{ and } \varphi'' > 0$$

- Suppose EN has no funds and a risk neutral investor provides the necessary I in exchange for a revenue contingent repayment $r(q)$.

Risky Debt As Optimal Contract-Innes (1990)

- Innes (1990) If the following two conditions are satisfied, then the debt contract is the optimal repayment contract $r(q)$.
  - 1) Two sided limited liability constraint $0 \leq r(q) \leq q$.
  - 2) A monotonicity constraint $0 \leq r'(q)$. (this can be easily justified)

Risky Debt As Optimal Contract-Innes (1990)

- Ignore the monotonicity constraint for the time being. The constrained optimization problem of the EN is as follows:

$$\underset{\{r(q),a\}}{Max} \int^{\bar{q}} [q - r(q)] f(q \mid a) dq - \varphi(a) - I \text{ subject to}$$

$$\int^{\bar{q}} [q - r(q)] f_a(q \mid a) dq = \varphi'(a) \qquad \text{(IC)}$$

$$\int^{\bar{q}} r(q) f(q \mid a) dq = I \qquad \text{(IR)}$$

$$0 \leq r(q) \leq q \qquad \text{(LL)}$$

Risky Debt As Optimal Contract-Innes (1990)

- The Lagrangian is

$$\int r(q)\left[\lambda - \frac{\mu f_a(q \mid a)}{f(q \mid a)} - 1\right] f(q \mid a)dq$$
$$+ \int q\left[1 + \frac{\mu f_a(q \mid a)}{f(q \mid a)}\right] f(q \mid a)dq - \varphi(a) - \mu\varphi'(a) - \lambda I$$

which is linear in $r(q)$ for all q.

Risky Debt As Optimal Contract-Innes (1990)

- Therefore, provided that ICC is binding ($\mu > 0$), the optimal schedule is

$$r^*(q) = \left\{ \begin{array}{ll} q & \text{if } \lambda > 1 + \mu \frac{f_a(q|a)}{f(q|a)} \\ 0 & \text{if } \lambda < 1 + \mu \frac{f_a(q|a)}{f(q|a)} \end{array} \right\}$$

which implies that it is optimal to reward the EN for revenue outcomes such that the likelihood ratio

$$\underbrace{\frac{f_a(q \mid a)}{f(q \mid a)}}_{\uparrow \text{ in } q \text{ under MLRP}} > \frac{\lambda - 1}{\mu}$$

which implies there exists a revenue level Z such that

$$r^*(q) = \left\{ \begin{array}{ll} 0 & \text{if } q > Z \\ q & \text{if } q < Z \end{array} \right\} \text{ (But this is not monotonic)}$$

Risky Debt As Optimal Contract-Innes (1990)

- Once we impose the monotonicity constraint $0 \leq r'(q)$, we have the standard debt contract

$$r_D^*(q) = \left\{ \begin{array}{ll} D & \text{if } q > D \\ q & \text{if } q \leq D \end{array} \right\}$$

where D is the lowest value that solves the IRC

$$\int^D qf(q \mid a^*)dq + [1 - F(D \mid a^*)]D = I$$

and a^* solves the ICC

$$\int^{\bar{q}} (q - D)\, f_a(q \mid a^*)dq = \varphi'(a^*).$$