ECO 5341 Infinitely repeated games

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Infinitely Repeated Games

- Consider the following Prisoners Dilemma

	C	D
C	2, 2	-1, 3
D	3, -1	0, 0

- In the one-shot version, the unique NE is (D,D).
- Can we sustain the outcome (C,C) if this game is "infinitely" repeated?
- There is no final period. The game is repeated every period.

Three Key Aspects

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Three Key Aspects

- Histories
  - Ex: (C,C) (C,C) (C,C)................................(C,C)
  - Ex: (C,C) (D,C) (D,D)................................(D,D)
- Strategies are now action plans contingent on histories. A strategy must specify what the player will do after every possible history.
- Future payoffs are discounted.

Payoffs

Payoffs

- Each player has a discount factor $\delta \in (0, 1)$.
- Suppose starting today, a player receives an infinite sequence of payoffs

$$u_1, u_2, u_3, u_4 \ldots\ldots\ldots\ldots$$

- The present value of this payoff sequence is

$$u_1 + \delta u_2 + \delta^2 u_3 + \delta^3 u_4 + \ldots\ldots\ldots\ldots\ldots$$

- Example: In each period you receive a payoff of 2 forever

$$2 + 2\delta + 2\delta^2 + 2\delta^3 + \ldots\ldots\ldots\ldots$$

$$= 2(1 + \delta + \delta^2 + \delta^3 + \ldots\ldots\ldots\ldots)$$
$$= \frac{2}{1-\delta}$$

- Example: In each period you receive a payoff of 2 forever

$$2 + 2\delta + 2\delta^2 + 2\delta^3 + ...$$

$$= 2(1 + \delta + \delta^2 + \delta^3 + ...)$$

$$= \frac{2}{1-\delta}$$

because for $\delta \in (0,1)$

$$1 + \delta + \delta^2 + \delta^3 + ... = \frac{1}{1-\delta}$$

Infinitely Repeated Games

- **Example:** Consider the payoff sequence

$$2, 2, 3, 3, 3, ..., 3, 3, 3....$$

- The present value is

$$2 + 2\delta + 3\delta^2 + 3\delta^3 + 3\delta^4.............$$

Infinitely Repeated Games

- **Example:** Consider the payoff sequence

$$2, 2, 3, 3, 3, ..., 3, 3, 3....$$

- The present value is

$$2 + 2\delta + 3\delta^2 + 3\delta^3 + 3\delta^4 ....$$

$$= 2(1 + \delta) + 3\delta^2(1 + \delta + \delta^2 + \delta^3 + ...$$

- **Example:** Consider the payoff sequence

$$2, 2, 3, 3, 3, \ldots\ldots, 3, 3, 3\ldots.$$

- The present value is

$$2 + 2\delta + 3\delta^2 + 3\delta^3 + 3\delta^4 \ldots\ldots\ldots\ldots\ldots$$

$$= 2(1+\delta) + 3\delta^2(1 + \delta + \delta^2 + \delta^3 + \ldots\ldots\ldots\ldots\ldots$$

$$= 2(1+\delta) + \frac{3\delta^2}{1-\delta}$$

- **Example:** Consider the payoff sequence

$$2, 0, 2, 0, 2, 0, 2, 0..............$$

- The present value is

$$2 + 2\delta^2 + 2\delta^4 + 2\delta^6.............$$

$$= 2 + 2\delta^2(1 + \delta^2 + \delta^4 + \delta^6 + ...)$$

$$= 2 + \frac{2\delta^2}{1 - \delta^2}$$

Infinitely Repeated Games

- A strategy must specify what the player will do after every possible history.

Infinitely Repeated Games

- A strategy must specify what the player will do after every possible history.
- A strategy can be based on
  - the whole history so far
  - only a portion of the history so far
  - even only what happened in the last period.

Infinitely Repeated Games

- A history h_t in period t is a collection of all outcomes in the prior $t-1$ stages.
- Examples of possible histories at time t

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$$h_t = \underbrace{(C,C)}_{t=1}, \underbrace{(C,C)}_{t=2}, \underbrace{(C,C)}_{t=3}, \underbrace{(C,C)}_{t=4}, \underbrace{(C,C)}_{t=5} \ldots\ldots \underbrace{(C,C)}_{t-1}$$

$$h_t = \underbrace{(C,C)}_{t=1}, \underbrace{(C,C)}_{t=2}, \underbrace{(C,C)}_{t=3}, \underbrace{(C,C)}_{t=4}, \underbrace{(C,D)}_{t=5} \ldots\ldots \underbrace{(D,D)}_{t-1}$$

- **Example: Grim Trigger Strategy**

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  - Play C in the very first stage.
  - Continue to play C as long as everyone has always played C in the past.
  - Defect and play D otherwise

- **Example: Tit for Tat**

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  - Play C in the very first stage.
  - Play whatever your opponent played in the last period

Equilibria of Infinitely Repeated Games

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- There is no end period of the game. We cannot apply a backward induction type algorithm.
- We use One-Shot-Deviation Property to check whether a strategy profile is a SPE.

One-Shot-Deviation Property (OSD)

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- A strategy profile is a SPE of an infinitely repeated game if and only if no player can gain by changing her action after any history, keeping both the strategies of the other players and the remainder of her own strategy constant.
    - Take a history, for each player check if she has a profitable OSD from the proposed strategy.
    - Do that for every possible history.
    - If no player has a profitable OSD after any history, the proposed strategy is a SPE.

Infinitely Repeated Games

- Recall the Trigger Strategy
  - Play C in the very first stage.
  - Play C as long as everyone has always played C
  - Defect and play D otherwise
- Is the above strategy a SPE?

- There are two types of histories to consider for a Trigger Strategy (TS).
- **Fully Cooperative history:** Nobody has ever defected at any point in the past

$$h_t = \underbrace{(C, C)}_{t=1}, \underbrace{(C, C)}_{t=2}, \underbrace{(C, C)}_{t=3}, \underbrace{(C, C)}_{t=4}, \underbrace{(C, C)}_{t=5} .... \underbrace{(C, C)}_{t-1}$$

After such a history, TS says play C in period t.

Infinitely Repeated Games

- Histories with **some defection in the past:** Somebody has played D at some point

$$h_t = \underbrace{(C,C)}_{t=1}, \underbrace{(C,C)}_{t=2}, \underbrace{(C,C)}_{t=3}, \underbrace{(C,D)}_{t=4}, \underbrace{(D,D)}_{t=5} .... \underbrace{(D,D)}_{t-1}$$

After such a history, TS says plays D in period t.

Infinitely Repeated Games

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- To check whether TS is a SPE, we need to check if there is any profitable One-Shot Deviation (OSD) for any player after each possible history.
- So we check if there is a profitable OSD from TS after a cooperative history and also after a history that includes defection by any player.

- **What do to after a history**

$$h_t = \underbrace{(C,C)}_{t=1}, \underbrace{(C,C)}_{t=2}, \underbrace{(C,C)}_{t=3}, \underbrace{(C,C)}_{t=4}, \underbrace{(C,C)}_{t=5} \ldots\ldots \underbrace{(C,C)}_{t-1}$$

- If you follow the TS and play C, you receive 2 forever.

$$2 + 2\delta + 2\delta^2 + 2\delta^3 + \ldots\ldots\ldots\ldots = \frac{2}{1-\delta}$$

- If you play D, you get 3 today and 0 from tomorrow onwards.

$$3 + 0\delta + 0\delta^2 + 0\delta^3 + \ldots\ldots\ldots\ldots = 3$$

Following TS optimal if

$$\frac{2}{1-\delta} \geq 3 \Rightarrow \delta \geq \frac{1}{3}.$$

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- **What do to after a history**

$$h_t = \underbrace{(C,C)}_{t=1}, \underbrace{(C,C)}_{t=2}, \underbrace{(C,C)}_{t=3}, \underbrace{(C,C)}_{t=4}, \underbrace{(C,C)}_{t=5} \ldots\ldots \underbrace{(C,D)}_{t-1}$$

- Following TS and playing D gets you 0 forever.
- If you deviate from TS and play C, you get -1 today and 0 starting from tomorrow (Why?).
- Hence after any history with D, following TS and playing D is optimal.

Infinitely Repeated Games

- Conclusion: Trigger Strategy is a SPE if and only if $\delta \geq \frac{1}{3}$.
- Cooperation can be sustained if players are patient enough and future interaction is likely enough.