Notes on Normal Form, Strict Dominance and Nash Equilibrium

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Normal Form Representation of a Game

- In the normal-form representation of a game, each player simultaneously chooses a strategy and the combination of strategies chosen by players determines a payoff for each player.
- Three ingredients
  - Set of Players
  - Set of Strategies Available to Each Player
  - A Payoff Function that assigns a payoff to each player for every possible strategy profile

Normal Form Representation

Prisoners' Dilemma

P2 (Column Player)

P1 (Row Player)

	D	C
D	2,2	10,0
C	0,10	8,8

- D refers to "Defect" and C refers to "Cooperate"
- Set of Players P1 and P2
- Set of Strategies $S_i = \{D, C\}$ for $i = 1, 2$

Normal Form Representation

A payoff function that assigns a payoff to each player for every possible strategy profile

P2

P1

	D	C
D	2,2	10,0
C	0,10	8,8

$u_1(D, D) = 2$		$u_2(D, D) = 2$
$u_1(D, C) = 10$		$u_2(D, C) = 0$
$u_1(C, D) = 0$		$u_2(C, D) = 10$
$u_1(C, C) = 8$		$u_2(C, C) = 8$

Strict Dominance

What is so special about the Prisoners' Dilemma?

P2

P1

	D	C
D	2,2	10,0
C	0,10	8,8

- (C,C) most socially desirable outcome.
- Can players achieve (C,C) as an "equilibrium" outcome when they choose actions to maximize own payoff?
- Based on a simple "rationality requirement" the answer is no.

Strict Dominance

Rationality Requirement: A rational player never uses a strictly dominated strategy

	D	C
D	2,2	10,0
C	0,10	8,8

$$u_1(D, C) = 10 > u_1(C, C) = 8$$
$$u_1(D, D) = 2 > u_1(C, D) = 0$$

- Regardless of what P2 does, playing D is strictly better than playing C for P1.We say that D strictly dominates C for P1.
- D strictly dominates C for P2 as well.

Definition

Definition of Strict Dominance in a 2-Player Game

- Consider two actions x and y for P1.
- Let $a_2 \in S_2$ denote a generic action in action set S_2 for P2.
- We say that x strictly dominates y for P1 if

$$u_1(x, a_2) > u_1(y, a_2) \text{ for every } a_2 \in S_2.$$

and denote the strict dominance relationship with

$$x \succ y.$$

Strict Dominance

	L	C	R
T	8,4	6,2	7,3
M	10,6	3,5	0,4
B	4,8	5,6	5,7

- Observe that $L \succ C$ because

$$u_2(T, L) = 4 > u_2(T, C) = 2$$
$$u_2(M, L) = 6 > u_2(M, C) = 5$$
$$u_2(B, L) = 8 > u_2(B, C) = 6.$$

- Similarly, verify that $L \succ R$. Since L strictly dominates every other strategy, we call L as **the strictly dominant strategy** for P2.

Strict Dominance

Another example

	L	C	R
T	8,4	3,5	4,3
M	10,6	6,7	5,4
B	4,4	5,6	-2,4

- Verify formally that $C \succ R$ and $C \succ L$. Hence C is a strictly dominant strategy for P2.
- Verify formally that $M \succ T$ and $M \succ B$. Hence M is a strictly dominant strategy for P1.
- We refer to (M,C) a strictly dominant strategy equilibrium.

Strict Dominance

By definition, a strictly dominant strategy is the unique "best response".

	L	C	R
T	8,4	3,5	4,3
M	10,6	6,7	5,4
B	4,4	5,6	-2,4

- Observe that

$$BR_1(L) = M \quad BR_1(C) = M \quad BR_1(R) = M$$

$$BR_2(T) = C \quad BR_2(M) = C \quad BR_2(B) = C$$

Nash Equilibrium

	L	M	R
U	8,4	6,5	4,5
D	10,6	6,7	5,4

- In the above example, D weakly dominates U. We denote this as $D \succsim U$.
- Also observe that $M \succ L$ and $M \succsim R$.
- We refer to (D,M) as a weakly dominant strategy equilibrium.

Nash Equilibrium

- Many games are not dominance solvable in the sense that players do not have a strictly or weakly dominant strategy.
- Hence, we require a stronger solution concept that produces tighter predictions.

Definition of Nash Equilibrium (NE) in a 2-Player Game

- We say that a strategy profile (x^*, y^*) is a Nash Equilibrium if

$$x^* \in BR_1(y^*)$$

and

$$y^* \in BR_2(x^*).$$

- We only require x^* to be a best response to y^* not the unique best response.
- Similarly we only require y^* to be a best response to x^* not the best response.

Nash Equilibrium

	L	R
U	2,1	2,1
D	0,0	2,0

- (U,L) is a NE because $U = BR_1(L)$ and $L \in BR_2(U)$.
- Can you find other NE?

Nash Equilibrium

Observation: If (x^*, y^*) is a strictly dominant strategy equilibrium, then it is also a unique NE

- Proof: If (x^*, y^*) is a strictly dominant strategy equilibrium, then x^* is strictly dominant for P1. This implies that

$$u_1(x^*, y^*) > u_1(a_1, y^*) \text{ for every } a_1 \in S_1 \Rightarrow x^* = BR_1(y^*).$$

- Similarly, y^* is strictly dominant for P2. This implies that

$$u_2(x^*, y^*) > u_1(x^*, a_2) \text{ for every } a_2 \in S_2 \Rightarrow y^* = BR_2(x^*).$$

- But then if $x^* = BR_1(y^*)$ and $y^* = BR_2(x^*)$, we conclude that (x^*, y^*) is a unique NE.

Nash Equilibrium

	L	C	R
T	8,4	3,5	4,3
M	10,6	6,7	5,4
B	4,4	5,6	-2,4

- Verify that the strictly dominant strategy equilibrium (M,C) is the unique NE.

Nash Equilibrium

	L	M	R
U	8,4	6,5	4,5
D	10,6	6,7	5,4

- Verify that the weakly dominant strategy equilibrium (U,M) is also a NE but it is not a unique NE.

- Two research firms, Firm 1 and Firm 2 simultaneously choose how much time to spend on research to develop a new drug. Firm 1 chooses $x_1 \geq 0$ and Firm 2 chooses $x_2 \geq 0$.

Nash Equilibrium

- The two firms' payoff functions are given by

$$u_1(x_1, x_2) = \begin{array}{|l|l|} \hline 10 - x_1 & \text{if } x_1 > x_2 \\ \hline 5 - x_1 & \text{if } x_1 = x_2 \\ \hline -x_1 & \text{if } x_1 < x_2 \\ \hline \end{array}$$

$$u_2(x_1, x_2) = \begin{array}{|l|l|} \hline 10 - x_2 & \text{if } x_2 > x_1 \\ \hline 5 - x_2 & \text{if } x_1 = x_2 \\ \hline -x_1 & \text{if } x_1 < x_2 \\ \hline \end{array}$$

Nash Equilibrium

$$u_i(x_i, x_j) = \begin{array}{|l|l|} \hline 10 - x_i & \text{if } x_i > x_j \\ \hline 5 - x_i & \text{if } x_i = x_j \\ \hline -x_i & \text{if } x_i < x_j \\ \hline \end{array}$$

- Is the strategy pair $(x_1 = 5, x_2 = 5)$ a NE? No because

$$u_1(5,5) = 0 < u_1(6,5) = 4$$

Nash Equilibrium

$$u_i(x_i, x_j) = \begin{cases} 10 - x_i & \text{if } x_i > x_j \\ 5 - x_i & \text{if } x_i = x_j \\ -x_i & \text{if } x_i < x_j \end{cases}$$

- Is the strategy pair $(x_1 = 5, x_2 = 5)$ a NE? No because

$$u_1(5,5) = 0 < u_1(6,5) = 4$$

- To verify that a strategy profile is not a NE, it is sufficient to find a profitable devitiation from that profile for any of the two players.

Nash Equilibrium

$$u_i(x_i, x_j) = \begin{array}{|l|l|} \hline 10 - x_i & \text{if } x_i > x_j \\ \hline 5 - x_i & \text{if } x_i = x_j \\ \hline -x_i & \text{if } x_i < x_j \\ \hline \end{array}$$

- Consider any strategy profile $x_1 = x_2 = a$ where $0 < a < 5$. We have

$$u_1(a, a) = 5 - a < u_1(5, a) = 5.$$

  Hence, no strategy profile $x_1 = x_2 = a$ where $0 < a < 5$ is a NE.

Nash Equilibrium

$$u_i(x_i, x_j) = \begin{cases} 10 - x_i & \text{if } x_i > x_j \\ 5 - x_i & \text{if } x_i = x_j \\ -x_i & \text{if } x_i < x_j \end{cases}$$

- Now consider $x_1 = x_2 = 0$. We have

$$u_1(0,0) = 5 < u_1(1,0) = 9$$

Hence $x_1 = x_2 = 0$ is not a NE either.

Nash Equilibrium

$$u_i(x_i, x_j) = \begin{array}{|c|c|} \hline 10 - x_i & \text{if } x_i > x_j \\ \hline 5 - x_i & \text{if } x_i = x_j \\ \hline -x_i & \text{if } x_i < x_j \\ \hline \end{array}$$

- Consider any strategy profile $x_1 = a$ and $x_2 = b$ where $0 < a < b < 5$. We have

$$u_1(a, b) = -a < u_1(0, b) = 0.$$

- Hence, there is also no NE with $x_1 = a$ and $x_2 = b$ where $0 < a < b < 5$.