2.6 Implicit Differentiation

\[ 3x^2 + y - 2 = 0 \]

**Implicit form**

\[ x + \sqrt{y} = 1 \]

**Explicit form**

\[ y = 2 - 3x^2 \]

\[ \frac{dy}{dx} = 1 - x \]

**y cannot be negative**

\[ D: (7 - \infty, 1) \]

Implicit Differentiation must be used to find the derivative when you do not have explicit form. Take the derivative of both sides of the equation with respect to \( x \) and then solve for \( \frac{dy}{dx} \) or \( y' \).

\[ 3x^2 + y - 2 = 0 \]

\[ \frac{d}{dx}(3x^2 + y - 2) = \frac{d}{dx}(0) \]

\[ 6x \frac{dx}{dx} + 1 \frac{dy}{dx} = 0 \]

\[ 6x + \frac{dy}{dx} = 0 \]

\[ \frac{dy}{dx} = -6x \]

\[ xy - 3x + 2x^2 = S \]

\[ \frac{d}{dx}(xy - 3x + 2x^2) = \frac{d}{dx}(S) \]

\[ x \frac{dy}{dx} + (x \frac{dx}{dx}) - 3 \frac{dx}{dx} + 4x \frac{dy}{dx} = 0 \]

\[ x \frac{dy}{dx} = -y + 3 - 4x \]

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These two answers appear different, but are consistent.

\[ \frac{dy}{dx} = -y + 3 - 4x \]

\[ = -(5x^{-1} + 3 - 2x) \]

\[ = -(5x^{-1} + 3 - 2x)(x^{-1} + 3x^{-1} - 4) \]

\[ = -5x^{-2} - 3x^{-1} + 2 + 3x^{-1} - 4 \]

\[ \frac{dy}{dx} = -5x^{-2} - 2 \] → same result as explicit form
In some equations, solving for $y$ in order to use explicit form is too complicated. Just easier to use implicit method.

Find $dy/dx$ implicitly: $y \sin x^2 = x \sin y^2$

\[ \frac{d}{dx} (y \sin x^2) = \frac{d}{dx} (x \sin y^2) \]

\[ y \cdot \cos x^2 (\sin x^2) + \frac{dy}{dx} \sin x^2 = x \cdot \cos y^2 (\sin y^2) + \frac{dy}{dx} \sin y^2 \]

\[ \frac{dy}{dx} (\sin x^2 - \cos y^2) = \sin y^2 - \cos y^2 \sin x^2 \]

\[ \frac{dy}{dx} = \frac{\sin y^2 - \cos y^2 \sin x^2}{\cos x^2 - \sin y^2} \]

Find $y''$ for $x^4 + y'' = 10$

$y': 4x^3 x' + 4y^3 y' = 0$

\[ \frac{4y^3 y'}{4y^3} = -x^3 \quad \rightarrow \quad y' = -x^3 y^3 \]

$y'': y'' = \frac{y^3 (-3x^2 x') - (-x^3)(3y^2 y')}{(y^3)^2}$

\[ y'' = -3x^2 y^3 + 3x^3 \sin y \quad \rightarrow \quad y'' = -x^3 y^3 \]

$y'' = -3x^2 y^3 + 3x^3 y' (-x^3 y^3)$

\[ y'' = -3x^2 y^3 - 3x^3 y' \quad \rightarrow \quad y'' = -3x^2 y^3 - 3x^3 y' \]

\[ y'' = \frac{-48x^6}{y^3} \]

Application

Find equation of tangent line(s) at $x = 1$

$x^2 + y^2 - xy - 7 = 0$

$x = 1$: $x^2 + y^2 - y - 7 = 0$

$\frac{2x}{2} \cdot x' + ay - (x \cdot y' + x' \cdot y) = 0$

$2x + ay - y' - y = 0$

$2x + ay - y' - y = 0$

$y' = y - ay - 2$

$\frac{y'}{y} = y - 2$

$m = \frac{y' (1, 3)}{1 - 2} = \frac{3 - 2}{1 - 2} = \frac{1}{-1}$

$m = \frac{4/5}{2/5} = \frac{4}{2}$

$m = \frac{4}{5}

y = \frac{4}{5} x + 2$
2.7 Related Rates

If \( a(t) \) and \( b(t) \) are both functions of time and they are related by some equation, then by implicit differentiation their derivatives \( a'(t) \) and \( b'(t) \) are also related.

When one rate is given, it can be used to find the other rate.

Related rate problems involve relating the derivatives of two or more quantities that change over time.

1) A 13 ft. tall ladder is leaning against the side of a building. If the bottom of the ladder slides away from the wall at 3 ft/sec, how fast is the top of the ladder moving down the wall when the bottom is 5 ft. from the wall?

\[ x^2 + y^2 = 169 \]

\[ \frac{dx}{dt} = 3 \text{ ft/sec} \]

\[ \frac{dy}{dt} = \frac{-20}{\sqrt{80}} = -\frac{20\sqrt{2}}{8} \text{ ft/sec} \]

The ladder is moving down the side of the building at the rate of \( \frac{20\sqrt{2}}{8} \) ft/sec.

2) An observer, 500 meters from the launch pad of a rocket, watches it ascend vertically at 100 m/sec.

a) Find the rate of change of the distance between the rocket and the observer when the rocket is 400 m high.

\[ y^2 = x^2 + 500^2 \]

\[ \frac{dy}{dt} = 2ax \frac{dx}{dt} \]

\[ 2(500) \frac{dy}{dt} = 2(400)(100) \]

\[ \frac{dy}{dt} = 48 \text{ m/sec} \]

The distance between the observer and the rocket is increasing at 48 m/sec.
b) Find the rate of change of the angle between the ground and the observer's line of sight when the rocket is 400 m high.

\[ \tan \theta = \frac{x}{200} \]

\[ 300 \tan \theta = x \]

\[ 300 \tan \theta = 400 \]

\[ \tan \theta = \frac{400}{300} = \frac{4}{3} \]

\[ \sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{dx}{dt} \]

\[ 300 \left( \frac{4}{3} \right)^2 \cdot \frac{d\theta}{dt} = 60 \]

\[ 300 \left( \frac{4}{3} \right)^2 \cdot \frac{d\theta}{dt} = 60 \]

\[ \frac{100}{9} \cdot \frac{d\theta}{dt} = 60 \]

\[ \frac{d\theta}{dt} = \frac{27}{5} \text{ radians/sec} \]

Angle is increasing by \( \frac{27}{5} \) radians/sec.
28 Linear Approximations and Differentials

Linear Approximation of a Function Value:

The tangent line to a function \( y = f(x) \) at a point \( a \) can be used to estimate a value for \( f(x) \) when \( x \) is near \( a \).

Let \( L(x) \) be the equation of the tangent line to \( y = f(x) \) at \( x = a \).

\[ L(x) = f(a) + f'(a)(x-a) \rightarrow \text{Linearization of } f + \epsilon 
\]

So \( f'(x) \approx f'(a) + f''(a)(x-a) \)

1) Find the linear approximation.

\[ f(x) = 5x^3 + 6x \text{ at } x = 2 \]

\[ f(2) = 5(2)^3 + 6(2) = 52+12 = 52, \quad f'(2) = 15(2)^2 + 6 = 60 + 6 = 66 \]

Since \( f(x) \approx L(x) \)

\[ 5x^3 + 6x \approx 52 + 66(x-2) \]

\( L(1.98) = 52 + 66(1.98-2) = 52 + 66(-.02) \approx 50.16 \)

\[ f(1.98) \approx 50.16 \text{ (from a calculator)} \]

\[ f(1.98) = 5(1.98)^3 + 6(1.98) = 50.19 \]

2) Approximate \( f(x) \) using linear approximation.

Choose \( f(x) = \sqrt{x} \) with \( x = 64 \)

\[ f(64) = \sqrt{64} = 8, \quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \]

\[ f'(64) = \frac{1}{2\sqrt{64}} = \frac{1}{16} \]

\[ L(x) = f(64) + f'(64)(x-64) \]

\[ L(x) = 8 + \frac{1}{16}(x-64) \rightarrow \text{since } f(x) \approx L(x) \]

\[ \sqrt{x} \approx 8 + \frac{1}{16}(x-64) \]

\[ L(60.01) = 8 + \frac{1}{16}(60.01-64) = 8 + 0.125 = 8.125 \]

\[ f(60.01) \approx 8.125 \text{ (calculator: } f(60.01) = \sqrt{60.01} = 8.01240) \]
3. Find the linear approximation of \( f(x) = \sin x \) at \( x = 0 \).

Use it to find \( f(\frac{\pi}{4}) \).

\[
\begin{align*}
f(0) &= \sin 0 = 0 \\
f'(x) &= \cos x \\
f'(0) &= \cos 0 = 1
\end{align*}
\]

\[
L(x) = f(0) + f'(0)(x - 0)
\]

\[
L(x) = 0 + 1(x) 
\]

Since \( f(x) \approx L(x) \)

\[
L(\frac{\pi}{4}) = \frac{\pi}{4} \approx 0.2094395
\]

\[
f(\frac{\pi}{4}) \approx 0.2094395
\]

(calculated: \( f(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = 0.20791169 \))

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**Differential approximations using differentials**

Let \( y = f(x) \) be a differentiable function.

a) The differential \( dx \) is an independent variable.

b) The differential \( dy \) is defined as \( dy = f'(x)dx \)

If we let \( dx = \Delta x \), then for small values of \( dx \), the change in the function \( dy \) is approximately the same as the change in the function.

\[
\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}
\]

This is handy since \( dy \) may be easier to compute than \( \Delta y \).

\[
dy \approx \Delta y \text{ when } \Delta x \text{ is small}
\]

\( dy \) may be thought of as the error in calculating a value for \( y \) if an error of \( \Delta x \) is made in estimating \( x \).

\[
\frac{dy}{y} \text{ is the relative error}
\]

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1) Compute \( dy \) & \( \Delta y \) for \( f(x) = x^2 + 3x + 2 \) at \( x = 2 \) with \( \Delta x = dx = 0.1 \)

\[
\begin{align*}
f'(x) &= 2x + 3 \\
f'(2) &= 2(2) + 3 = 7 \\
dy &= f'(2)dx \\
&= 7 \cdot 0.1 \\
dy &= 0.7
\end{align*}
\]

\[
\begin{align*}
f(2) &= 2^2 + 3(2) + 2 = 10 \\
f(2.1) &= 2.1^2 + 3(2.1) + 2 = 10.71 \\
\Delta y &= f(2.1) - f(2) \\
&= 10.71 - 10 \\
&= 0.71
\end{align*}
\]

\( dy \) & \( \Delta y \) are close, but \( dy \) is easier to calculate.
2) The sides of a square field are measured and found to be 80 m with a possible error of 0.02 m. Calculate the area: $A = x^2$

\[ A = 80^2 \]
\[ A = 6400 \text{ m}^2 \]

Estimate the maximum error in this calculation.

Let $x =$ side of the field

\[ A = \text{area} \]

Given: $\Delta x = dx = 0.02$ when $x = 80$

\[ A = x^2 \]
\[ dA = 2x \, dx \]
\[ dA = 2(80)(0.02) = 3.2 \text{ m}^2 \text{ max error} \]

\[ \frac{dA}{A} = \frac{3.2 \text{ m}^2}{6400 \text{ m}^2} = 0.0005 \text{ or } 0.05\% \text{ relative error} \]