Chapter 2: DERIVATIVES

2.1 DERIVATIVES AND RATES OF CHANGE

TANGENTS TO THE GRAPH OF \( f(x) \)

1) Find the equation of the line tangent to \( f(x) = 3x^2 - 5x \) at \( (3, 2) \)

Two ways to find the slope:

a) \( m = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} \)

\[
m = \lim_{x \to 2} \frac{3x^2 - 5x - 3(2)^2 + 5(2)}{x - 2} = \lim_{x \to 2} \frac{3x^2 - 5x - 12 + 10}{x - 2} = \lim_{x \to 2} \frac{3x^2 - 5x - 2}{x - 2} = \lim_{x \to 2} \frac{(3x + 1)(x - 2)}{x - 2} = 3(2) + 1 = 7
\]

\[m = 7\]

\[y = 7(x - 2)\]

b) \( m = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} \)

\[
m = \lim_{h \to 0} \frac{3(2 + h)^2 - 5(2 + h) - 3(2)^2 + 5(2)}{h}
= \lim_{h \to 0} \frac{12h + 12h^2 + 3h - 10 - 5h - 10}{h}
= \lim_{h \to 0} \frac{3(2h + 3h)}{h} = 7 + 3(0) = 7
\]

\[m = 7\]

\[y = 7(x - 2)\]

or

\[y = 7x - 12\]

VELOCITY AND RATES OF CHANGE

AVERAGE VELOCITY FROM TIME \( t = a \) TO \( t = a + h \) GIVEN BY

\[
V_{ave} = \frac{f(a + h) - f(a)}{h}
\]

ALSO CALLED "AVERAGE RATE OF CHANGE"

INTEGRAL VELOCITY (OR JUST VELOCITY) AT TIME \( t = a \) GIVEN BY:

\[
v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

ALSO CALLED INSTANTANEOUS RATE OF CHANGE OR SIMPLY "RATE OF CHANGE"
Example: The displacement (in m) of a particle moving in a straight line is \( s(t) = t^2 - 8t + 18 \) (t is in seconds).

a) Find average velocity over \([3, 4]\) \( a = 3 \), \( a + h = 4 \) \( h = 1 \)

\[
V_{ave} = \frac{f(a + h) - f(a)}{h} = \frac{f(4) - f(3)}{1} = \frac{[4^2 - 8(4) + 18] - [3^2 - 8(3) + 18]}{1} = -1 \text{ m/sec}
\]

b) Find the instantaneous velocity at \( t = 3 \)

\[
V(3) = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{(3 + h)^2 - 8(3 + h) + 18}{h} - \frac{3^2 - 8(3) + 18}{h}
\]

\[
= \lim_{h \to 0} \frac{9 + 6h + h^2 - 24 - 8h + 18}{h} = \lim_{h \to 0} \frac{h^2 - 2h}{h} = -2 + 0 = -2 \text{ m/sec}
\]

The derivative: If \( y = f(x) \), then the derivative of \( f \) at \( x = a \) is denoted by \( f'(a) \).

Two formulas:

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

The derivative is used for:

a) Slope of tangent line
b) Instantaneous velocity
c) Instantaneous rate of change

def \( f'(a) \) for \( f(x) = \sqrt{x + 1} \)

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{\sqrt{a + h + 1} - \sqrt{a + 1}}{h}
\]

\[
= \lim_{h \to 0} \frac{(a + h + 1) - (a + 1)}{h(\sqrt{a + h + 1} + \sqrt{a + 1})} = \lim_{h \to 0} \frac{2h}{h(\sqrt{a + h + 1} + \sqrt{a + 1})} = \frac{2}{\sqrt{a + 1} + \sqrt{a + 1}}
\]

\[
f'(a) = \frac{2}{2\sqrt{a + 1}}
\]
If \( f(x) = \frac{5x}{1+x^2} \), find \( f'(x) \) and write the equation of the tangent line to \( f(x) \) at \( (2, a) \).

\[
f'(x) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}\]

\[
= \lim_{x \to 2} \left( \frac{\frac{5x}{1+x^2} - \frac{5(2)}{1+2^2}}{x - 2} \right)
\]

\[
= \lim_{x \to 2} \left( \frac{5x - 10}{1+x^2} \right) \cdot \frac{1}{1+x^2}
\]

\[
= \lim_{x \to 2} \frac{5x - 10}{(x-2)(1+x^2)} = \lim_{x \to 2} \frac{5x - 2x^2}{(x-2)(1+x^2)}
\]

\[
= \lim_{x \to 2} \frac{-2x^2+5x-10}{x-2} = \lim_{x \to 2} \frac{-2(x-2)(x-2.5)}{x-2} = -\frac{1}{4}
\]

\[
m = -\frac{1}{4} \quad (2, a) \rightarrow \quad y - a = -\frac{1}{4} (x - 2)\]
2.2 The derivative as a function.

2.1 gave us \( f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \), now we replace \( a \) by \( x \) which gives us a new function called the derivative of \( f \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

1) Find \( f'(x) \).

\[
f(x) = x^3 - 2x^2 + 5
\]

\[
f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - 2(x+h)^2 + 5 - (x^3 - 2x^2 + 5)}{h}
\]

\[
= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 2x^3 - 4x^2h - 2xh^2 + 10) - (x^3 - 2x^2 + 5)}{h}
\]

\[
= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 4xh - 2h}{h}
\]

\[
= \lim_{h \to 0} (3x^2x + 3xh + h^2 - 4x - 2)
\]

\[
f'(x) = 3x^2 - 4x
\]

BRIEF & NOTATION

1) \( f \) is differentiable at \( x \) means that \( f'(x) \) exists.

2) \( f \) is differentiable on an interval if \( f \) is differentiable at every point in the interval.

3) Notations for the derivative of \( f \):

\[
f'(x) \quad y' \quad \frac{df}{dx} \quad \frac{du}{dx} \quad \frac{d}{dx}(f(x)) \quad Df(x) \quad DXf(x)
\]

4) If \( f \) is differentiable at point \( a \), then \( f \) is continuous at \( a \).

* But continuity does not imply differentiability.

Where is \( y = f(x) \) not differentiable?

At: \( x = -2 \) → Sharp corner

\( x = 0, 1 \) → Discontinuous

\( x = 3 \) → Vert. tangent
2. Basic Differentiation Formulas

Derivative Rules (or short cuts)

If \( y = f(x) \), then \( f'(x), y', \frac{dy}{dx}, \frac{d}{dx}(y), \frac{d}{dx}(f(x)) \)
all represent the derivative of \( f(x) \).

**Constant Function:** \( f(x) = c \), then \( f'(x) = 0 \)

- a) \( f(x) = 3 \) \( \Rightarrow f'(x) = 0 \)
- b) \( y = -5 \) \( \Rightarrow y' = 0 \)
- c) \( \frac{d}{dx}(3.2) = 0 \)

**Linear Function:** \( f(x) = x \), then \( f'(x) = 1 \)

**Power Rule:** \( f(x) = x^n \), then \( f'(x) = nx^{n-1} \)

- a) \( f(x) = x^3 \) \( \Rightarrow f'(x) = 3x^2 \)
- b) \( f(x) = \sqrt[3]{x^2} \) \( \Rightarrow f'(x) = \frac{2}{3}x^{-\frac{1}{3}} \)
- c) \( y = \frac{1}{x} \) \( \Rightarrow y' = -\frac{1}{x^2} \)

**Constant Multipliers:** \( y = af(x) \), then \( \frac{dy}{dx} = af'(x) \)

- a) \( f(x) = 4x^5 \) \( \Rightarrow f'(x) = 20x^4 \)
- b) \( y = \frac{2}{3}x^3 \) \( \Rightarrow y' = 2x^2 \)

**Sum/Difference:** \( y = f(x) \pm g(x) \), then \( y' = f'(x) \pm g'(x) \)

- a) \( f(x) = 3x^4 - 2x^3 + x^2 - 5x + 7 \) \( \Rightarrow f'(x) = 12x^3 - 6x^2 + 2x - 5 \)
- b) \( f(x) = 3 - \frac{7}{x^2} \) \( \Rightarrow f'(x) = 14x^{-3} \)
Trig Functions

\[ f(x) = \sin x, \text{ then } f'(x) = \cos x \]

\[ f(x) = \cos x, \text{ then } f'(x) = -\sin x \]

a) \[ y = \sin \frac{\pi}{2} \quad (y = 1) \]
   \[ y' = 0 \]

b) \[ f(x) = -3 \cos x \]
   \[ f'(x) = -3(-\sin x) = 3\sin x \]

f'(x) = 3\sin x

c) \[ g(x) = 5\sin x + 3\cos x \]
   \[ g'(x) = 5\cos x - 3\sin x \]

Higher Derivatives:

2nd derivative = derivative of \( f'(x) \)

If \( y = f(x) \), \( y' = f'(x) \), \( y'' = f''(x) \)

a) \[ f(x) = 2x^2 - 6x + 3\sin x + 5 \]
   \[ f'(x) = 4x - 6 + 3\cos x \]
   \[ f''(x) = 4 - 3\sin x \]

3rd derivative = derivative of \( f''(x) \)

\[ f'''(x) = -3\cos x \]

4th derivative = \( f''''(x) \)

5th derivative = \( f''''(x) \)

\[ \text{etc.} \]

Find 38th derivative of \( f(x) = \cos x \)

\[ f'(x) = -\sin x \]

\[ f''(x) = -\cos x \]

\[ f'''(x) = \sin x \]

\[ f''''(x) = \cos x \]

\[ \text{Pattern repeats every 4th der.} \]

\[ \frac{\pi}{2} \]

\[ f^{(28)}(x) = f^{(3)}(x) = -\cos x \]
Applications

1) Given: \( f(x) = x^4 - 8x^3 + 7 \), find the following

2) Equation of tangent line at \( x = -1 \).

   Slope: \( f'(x) = 4x^3 - 24x^2 \)
   
   \( m = f'(-1) = 4(-1)^3 - 24(-1)^2 = -4 - 24 = -28 \) \( m = -28 \)
   
   Point: \( f(-1) = (-1)^4 - 8(-1)^3 + 7 = 1 + 8 + 7 = 16 \) \((-1, 16)\)
   
   Line: \( y - 16 = -28(x + 1) \)

3) Equation of normal line at \( x = -1 \) (tangent line at \( P \))

   \( m = \frac{1}{28} \)
   
   Line: \( y - 16 = \frac{1}{28}(x + 1) \)

4) Find the values of \( x \) where the tangent is horizontal.

   \( m = 0 \rightarrow f'(x) = 0 \)
   
   \( 4x^3 - 24x^2 = 0 \)
   
   \( 4x^2(x - 6) = 0 \)
   
   \( x = 0, x = 6 \)

5) Given: \( f(x) = x + 2 \cos x \), find the values of \( x \) where \( f(x) \) has horizontal tangents over the interval \([0, 2\pi]\)

   \[ f'(x) = 1 - 2\sin x \]
   
   \[ 0 = 1 - 2\sin x \]
   
   \[ 2\sin x = 1 \]
   
   \[ \sin x = \frac{1}{2} \]
   
   \( x = \frac{\pi}{6}, \frac{5\pi}{6} \)

6) Water is flowing out of a water tower so that after \( t \) minutes there are \( 1000 - 10t - t^3 \) gallons remaining. How fast is the water flowing after 3 minutes?

   \[ V(t) = 1000 - 10t - t^3 \] Volume
   
   \[ V'(t) = -10 - 3t^2 \]
   
   \[ V'(3) = -10 - 3(3)^2 \]
   
   \[ = -32 \text{ gal/min} \]
4) A space shuttle is 110t + t³ meters from its launch pad t seconds after lift off. \( S(t) = 110t + t^3 \)

a) Find the velocity at time \( t = 3 \).

Velocity \( \Rightarrow \) 1st derivative

\[ V(t) = S'(t) = 110 + 3t^2 \]

\[ S'(3) = 110 + 3(3)^2 \]

\[ = 18 \text{ m/sec} \]

b) Find acceleration at time \( t = 3 \).

Acceleration \( \Rightarrow \) 2nd derivative

\[ S''(t) = V(t) \]

\[ S''(3) = (110 + 3(3)^2) \]

\[ = 18 \text{ m/sec}^2 \]
2.4. The Product and Quotient Rules

**Product Rule:** If \( y = f(x) \cdot g(x) \), then \( y' = f(x)g'(x) + f'(x)g(x) \)

1) \( f(x) = 5x^4 \quad f'(x) = 20x^3 \\
\)
\[
f(x) = 5x^4 + 20x^3 \\
f'(x) = 20x^3 + 40x^2 \\
f'(x) = 20x^2(2x + 1)
\]

2) \( f(x) = (x^3 + 1)(2x + 1) \\
\)
\[
f(x) = (x^3 + 2x^2 + x + 1) \\
f'(x) = 3x^2(2x + 1) + 3x + 1
\]

3) \( y = x^3 \sin x \\
\)
\[
y' = x^3 \cos x + 3x^2 \sin x \\
y' = x^2(3x \cos x + 2x^2 \sin x)
\]

**The Quotient Rule:** If \( y = \frac{f(x)}{g(x)} \), \( g(x) \neq 0 \), then \( y' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \)

1) \( f(x) = \frac{\sqrt{x^2 + 10}}{x^2 + 1} \\
\)
\[
f(x) = \frac{x^2 + 10}{(x^2 + 1)^2} \\
f'(x) = \frac{2x(x^2 + 1)}{(x^2 + 1)^2} \\
f'(x) = \frac{2x^3 + 2x}{(x^2 + 1)^2} \\
f'(x) = \frac{2x^2}{(x^2 + 1)^2}
\]

2) \( f(x) = \frac{1 + x}{x^2 - 1} \\
\)
\[
f(x) = \frac{x^2 - 1}{x^2 - 1} \\
f'(x) = \frac{1 + x}{(x^2 - 1)^2} \\
f'(x) = \frac{2x^2 - 2x - 1}{(x^2 - 1)^2}
\]

**Trig Functions:**

\[
\begin{align*}
\text{Show } & \frac{d}{dx} (\cot x) = -\csc^2 x \\
\text{Show } & \frac{d}{dx} (\sec x) = \sec x \tan x \\
\text{Show } & \frac{d}{dx} (\sin x) = \cos x \\
\text{Show } & \frac{d}{dx} (\cos x) = -\sin x \\
\text{Show } & \frac{d}{dx} (\tan x) = \sec^2 x \\
\text{Show } & \frac{d}{dx} (\cot x) = -\csc^2 x
\end{align*}
\]
Applications

Find the equation of the tangent line to

\[ y = (1+x) \cos x + (0, 1) \]

\[ y' = (1+x)(-\sin x) + (1) \cos x \]

\[ y' = -\sin x - x\sin x + \cos x \]

\[ m = -\sin 0 - 0 \cdot \sin 0 + \cos 0 \]

\[ m = 1 \]

\[ y - 1 = 1(x - 0) \]

\[ y - 1 = x \quad \text{or} \quad y = x + 1 \]

given: \( f(x) = \sec x \), \( f''(\frac{\pi}{4}) \)

\[ f'(x) = \sec x \tan x \]

\[ f''(x) = \sec x \cdot \sec^2 x + \sec x \tan x \cdot \tan x \]

\[ f''(x) = \sec^3 x + \sec x \tan^2 x \]

\[ f''(\frac{\pi}{4}) = \sec^3 (\frac{\pi}{4}) + \sec (\frac{\pi}{4}) \tan^2 (\frac{\pi}{4}) \]

\[ = \left( \sqrt{2} \right)^3 + \sqrt{2} \cdot 1 \]

\[ = 2\sqrt{2} + \sqrt{2} \]

\[ f''(\frac{\pi}{4}) = 3\sqrt{2} \]
2.5 The Chain Rule

**General Power Rule**: If $y = f(u)$, $u = g(x)$, and $y = (f \circ g)(x)$, then the Chain Rule for computing the derivative is:

$$y = (f \circ g)(x) = f(g(x))$$

$$y' = [f(g(x))]' = f'(g(x)) \cdot g'(x)$$

*(derivative of outside function times derivative of inside)*

1) $f(x) = (5x+2)^3$

$$f'(x) = 3(5x+2)^2 \cdot 5$$

2) $f(x) = \frac{3}{2x+4}^4 = \frac{3}{(x+3)^2}$

$$f'(x) = -6(2x+4)^{-3} \cdot 2$$

3) $f(x) = \sqrt{4-x} = (4-x)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}(4-x)^{-\frac{1}{2}} \cdot (-1)$$

Sometimes more than one function is necessary

1) $y = (3 + (x^3 - 2x)^5)^8$

$$y' = 8 \cdot (3 + (x^3 - 2x)^5)^7 \cdot (3(x^3 - 2x)^4 \cdot (3x^2 - 2))$$

2) $y = \tan^3 \frac{1}{x} = (\tan(\frac{1}{x}))^3$

$$y' = 3 \cdot (\tan(\frac{1}{x}))^2 \cdot (\sec^2(\frac{1}{x})) \cdot \frac{1}{x^2}$$

$$\sec^2 \frac{1}{x} = \frac{2}{x^2} \tan^2 \frac{1}{x} \sec^2 \frac{1}{x}$$

3) $f(x) = \sin(\cos(\tan(x)))$

$$f'(x) = \cos(\cos(\tan(x))) \cdot -\sin(\tan(x)) \cdot \sec^2 x$$

$$f'(x) = -\cos(\cos(\tan(x)) \sin(\tan(x)) \sec^2 x$$
Chains Rule & Products/Quotients

1) \( f(x) = \sin 2x \cos 3x \)  
   \( f'(x) = \sin 2x (-\sin 3x) + (\cos 3x) \cdot 2 \cdot \cos 2x \)  
   \( f'(x) = -3 \sin 2x \sin 3x + 2 \cos 3x \cos 2x \)

2) \( f(x) = (2x^2 - 2)^3 \)  
   \( f'(x) = (2x^2 - 2)^2 \cdot 4x + 2(2x^2 - 2) \cdot 4x \)  
   \( f'(x) = 2(2x^2 - 2) \cdot [x(2x^2 - 2) + 2(x^2 - 2)] \)  
   \( f'(x) = 2(x^2 + 4x^3 - 3x - 4) \)

3) \( f(x) = \frac{x^2 + 2x - 1}{(x+1)^2} \)  
   \( f'(x) = \frac{(x+1)^2 \cdot (2x+2) - (x^2 + 3x - 1) \cdot 2(x+1)}{(x+1)^4} \)  
   \( f'(x) = \frac{(x+1)(2x+2) - 2(x^2 + 3x - 1)}{(x+1)^3} \)  
   \( f'(x) = \frac{2x^2 + 6x + 2 - 2x^2 - 2x - 2}{(x+1)^3} \)  
   \( f'(x) = \frac{-x + 5}{(x+1)^3} \)

Applications

1) \( f(x) = \sqrt[3]{15 - 2x} = x \cdot (15 - 2x)^{-\frac{1}{3}} \)  
   Find equation of tangent line and normal line at \( x = 3 \)  
   \( \text{Find equation of tangent line at } x = 3 \)  
   \( x = 3 : f(3) = 3, f'(3) = \frac{1}{9}, q = 9, (3, 9) \)  
   \( f'(x) = \cdot \)  
   \( f'(x) = (15 - 2x)^{-\frac{1}{3}} \cdot [-(15 - 2x)] \)  
   \( f'(x) = (15 - 2x)^{-\frac{1}{3}} \cdot [15 - 2x] \)  
   \( m = f'(3) = \frac{15 - 9}{15 - 6} = \)  

   **Tangent:**  \( y - q = \frac{6}{3} (x - 3) \)  
   **Normal:**  \( y - q = -\frac{1}{6} (x - 3) \)
2) \( f(x) = \frac{x^3}{(3x-2)^2} \)

Find where tangent line is horizontal.

\[
f'(x) = \frac{(3x-2)^2(3x^2) - x^3 \cdot 2(3x-2)(3)}{(3x-2)^4}
\]

\[
= \frac{3x^2(3x-2) - 6x^3}{(3x-2)^3} = \frac{3x^2(3x-2) - 2x}{(3x-2)^4}
\]

Tangent is horizontal when \( f'(x) = 0 \)

\[
0 = \frac{3x^2(x-2)}{(3x-2)^3}
\]

\[
0 = 3x^2(x-2) \quad \rightarrow \quad x = 0, \quad x = 2
\]

3) The mean temperature at noon each day, \( x \), at the top of Mt. Arthur is approximated by:

\[
T(x) = 60 + 30 \sin \left( \frac{\pi(x-200)}{180} \right) \text{ degrees } F.
\]

Assume 360 days = 1 yr.

Find the rate of change of the daily temperature for January 15,

\[
T'(x) = 30 \cos \left( \frac{\pi(x-200)}{180} \right) \cdot \frac{\pi}{180}
\]

\[
T'(15) = \frac{\pi}{180} \cos \left( \frac{\pi(-15)}{180} \right)
\]

\[
\approx -0.5216 \text{ /day}
\]

On Jan 15, the noon temperature is decreasing by about ½ degree per day.