On the Feasible Payoff Set of Two-Player Repeated Games with Unequal Discounting*

Bo Chen† and Satoru Fujishige‡

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Abstract

We provide a novel characterization of the feasible payoff set of a general two-player repeated game with unequal discounting. In particular, we show that generically the Pareto frontier shifts outwards and the feasible payoff set expands in the sense of set inclusion, as the time horizon increases. This result reinforces and refines the insight in Lehrer and Pauzner (1999) by showing that a longer horizon enables the players to conduct intertemporal trade in a more flexible fashion.

Key Words: intertemporal trade, feasible payoff set, repeated game, unequal discounting; JEL Classification: C70, C72, C73.

1 Introduction

Existing literature on repeated games has mainly focused on repeated games where players have identical discount factors (see, e.g., Fudenberg and Maskin (1986)). In this setting, the feasible (discounted average) payoff set of the repeated game coincides with that of the stage game. With unequal discounting, however, these two payoff sets are typically different, as players can now arrange a path of play where the relatively impatient players receive more payoffs in early stages of the repeated game while the patient players obtain more later. Such intertemporal trade creates new cooperative possibilities for the players in the repeated game.

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†Department of Economics, Southern Methodist University (bochen@smu.edu).
‡Research Institute for Mathematical Sciences, Kyoto University (fujishig@kurims.kyoto-u.ac.jp).
In a seminal paper, Lehrer and Pauzner (1999) first formally study the effect of introducing unequal discounting on infinitely repeated games. They show that the feasible payoff set of a general two-player infinitely repeated game is larger than the convex hull of the stage-game payoffs, as a result of players’ being able to mutually benefit from trading payoffs across time. Following this insight, Chen (2007) uses a simple two-player $T$-period repeated game where the Pareto frontier of the stage game contains exactly two vertices to show that, fixing a given pair of discount factors and the time length between two consecutive stages being 1, the Pareto frontier of the $(T + 1)$-period game strictly dominates that of the $T$-period game, in the sense that no internal vertex of the Pareto frontier of the $(T + 1)$-period game touches that of the $T$-period game. A direct implication of this result is that, with proper payoff normalizations, a similar result holds for more general two-player finitely repeated games where the feasible payoff set of the stage game is an orthogonal triangle. In addition, the set of feasible payoffs of such games is also monotonic in the horizon of the game, in the set-inclusion sense.

In this paper, using Chen (2007) as building blocks, we develop a novel approach to characterize the feasible payoff set for a two-player (finitely) repeated game (denoted as $G^T$) with unequal discounting and a general stage payoff structure. We first represent the feasible payoff set of the stage game $G$ as the Minkowski sum (or the vector sum) of finitely many orthogonal triangles, each having the essential form of the example in Chen (2007). The feasible (discounted average) payoff set of the game $G^T$ can then be conveniently transformed into a double Minkowski sum of orthogonal triangles, where proper coordinate scaling is applied to each orthogonal triangle because of discounting. Given this representation, we characterize the vertices of the feasible payoff set of a generic $G^T$. Importantly, we find that each frontier of the feasible payoff set of $G^T$ shifts outwards as the horizon $T$ increases and consequently, the feasible payoff set gets enlarged monotonically over time. Our findings hence validate the intuition that the “gains” from intertemporal trade generally increase in the horizon of the repeated game.

This paper is only a technical contribution on characterizing feasible payoff sets for two-player repeated games with unequal discounting. Admittedly, a more important task in future research is to characterize equilibrium payoff sets for such games, as in the line of research on folk theorems for finitely repeated games (e.g., Benoît and Krishna (1985, 1987), Gossner (1995), Smith (1995) and González-Díaz (2006)).

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1 The authors also characterize the Nash equilibrium payoff set and the subgame perfect equilibrium payoff set for these games. They find that, in contrast to the classical folk theorem in Fudenberg and Maskin (1986), not all feasible individually rational payoffs can be supported by an equilibrium, even when both players are arbitrarily patient.

2 Gains are quoted here as intertemporal trade (generically) expands the feasible payoff set of the repeated game in all directions.
2 The Model

2.1 The Stage Game

Consider a strategic form game $G$ with an impatient player $I$ and a patient player $P$, where each player has a finite pure action set, $A_I$ and $A_P$. Assuming the existence of a public randomization device, let the set of correlated action profiles be $A$, denoting the set of all probability distributions over $A_I \times A_P$. The payoff functions are defined as $(X, Y) : A \to \mathbb{R}^2$, with $X(a)$ being player $I$’s payoff and $Y(a)$ being player $P$’s payoff from action profile $a \in A$.

Let the convex hull of the feasible payoff set of $G$ be $V = \text{co} \{(X(a), Y(a)) : a \in A\}$. Also denote the Pareto frontier of $V$ as $F$ and the collection of vertices of $F$ as $B$.

2.2 The Repeated Game

The repeated game $G^T$ is defined as the game with $G$ being repeated for $T$ times, where $T \in \mathbb{N}, T \geq 2$. Denote players’ discount factors as $\delta_I$ and $\delta_P$ with $1 > \delta_P > \delta_I > 0$. Given a payoff stream $\{(X_t, Y_t)\}_{t=1}^T$ where $(X_t, Y_t) \in V \ \forall t$, the players’ discounted average payoffs can be written as

$$\bar{U}_I^T = \frac{(1 - \delta_I)}{(1 - \delta_I^T)} \sum_{t=1}^T \delta_I^{t-1}X_t, \text{ and } \bar{U}_P^T = \frac{(1 - \delta_P)}{(1 - \delta_P^T)} \sum_{t=1}^T \delta_P^{t-1}Y_t.$$ 

Denote the set of feasible payoffs of $G^T$ as $V^T \subseteq \mathbb{R}^2$. Given the existence of a public randomization device, $V^T$ is a convex set, implying that the boundary of $V^T$ is a collection of points, each of which is obtained as a maximizer of some weighted sum of the players discounted average payoffs in $G^T$. For $t = 1, \ldots, T$, define

$$\gamma_I^T(t) = \frac{\delta_I^{t-1}}{1 + \delta_I + \ldots + \delta_I^{T-t}},$$
$$\gamma_P^T(t) = \frac{\delta_P^{t-1}}{1 + \delta_P + \ldots + \delta_P^{T-t}}.$$  

Then by definition, we have

$$V^T = \left\{ (Z_I, Z_P) \mid Z_I = \sum_{t=1}^T \gamma_I^T(t)X_t, Z_P = \sum_{t=1}^T \gamma_P^T(t)Y_t, (X_t, Y_t) \in V, \ \forall t = 1, \ldots, T \right\}.$$ 

For any set $W \subseteq \mathbb{R}^2$, define the coordinate scaling of $W$ by positive scalars $\beta_1$ and $\beta_2$ as

$$W[\beta_1, \beta_2] = \{(\beta_1X, \beta_2Y) \mid (X, Y) \in W\}.$$ 

Then it is easy to see that the feasible payoff set $V^T$ is alternatively expressed as

$$V^T = \sum_{t=1}^T V[\gamma_I^T(t), \gamma_P^T(t)].$$

\footnote{A brief discussion on the assumption of public randomization is provided in Section 3.2.}
where the sum is the vector sum or the Minkowski sum. Here the Minkowski sum of two sets \( W_1, W_2 \subseteq \mathbb{R}^2 \) is defined by

\[
W_1 + W_2 = \{(X_1 + X_2, Y_1 + Y_2) \mid (X_1, Y_1) \in W_1, (X_2, Y_2) \in W_2\}.
\]

The Minkowski sum of more than two sets is defined similarly. Notice in particular that the Minkowski sum is commutative and associative.

### 3 Results

Chen (2007) characterizes \( V^T \) for a game \( G^T \) where \( F \) contains only two vertices \( B = \{(1, 0), (0, 1)\} \). With this feature, the optimal path of play, yielding an (interior) vertex of the Pareto frontier of \( V^T \), takes a simple form where the players first play the pure-action profile corresponding to \((1, 0)\) for the first \( k \) periods, \( k \in \{1, \ldots, T-1\} \), and then switch to the other action profile for \((0, 1)\) for the rest of the game. Such a dichotomous property enables one to express the Pareto frontier \( F^T \) of \( V^T \) explicitly.

Given \( \theta \in (0, 1) \) and \( y = \left[ \ln \left( \frac{1 - \theta(1 - \delta_I^T)}{\ln \delta_I} \right) \right] \), we have

\[
F^T = \left\{ (U_I^T, \bar{U}_P^T) \mid U_I^T = \theta, \bar{U}_P^T = \frac{\delta_P^y - \delta_P^T}{1 - \delta_P^T} - \frac{\delta_P^y(1 - \delta_P)}{\delta_I^y(1 - \delta_I)} \left( \frac{\theta(1 - \delta_I^T) - (1 - \delta_P^y)}{1 - \delta_P^T} \right) \right\}.
\]

This explicit form then allows one to establish \( U_I^{T+1} > \bar{U}_P^T, \forall q, T \) by direct computation. In the sequel, we first recast the results of Chen (2007) using our current notation and then extend these results to general two-player games.

First, denote by \( S_{p,q} \) the simplex (or the orthogonal triangle) with three vertices \((0, 0), (p, 0)\) and \((0, q)\) for positive real numbers \( p \) and \( q \). Then from (2), we have

\[
S_{p,q}[\beta_1, \beta_2] = S_{\beta_1 p, \beta_2 q}.
\]

It then follows from (3) and (4) that if the feasible payoff set \( V \) takes the form of \( S_{p,q} \), then the feasible payoff set \( V^T \) can be expressed as

\[
S_{p,q}^T = \sum_{t=1}^{T} S_{p,q}[\gamma_I^T(t), \gamma_P^T(t)] = \sum_{t=1}^{T} S_{\gamma_I^T(t)p, \gamma_P^T(t)q}.
\]

Chen (2007) provides a complete characterization of \( S_{p,q}^T \). In particular, he shows that

\[
S_{p,q} = S_{p,q}^1 \subset S_{p,q}^2 \subset \ldots \subset S_{p,q}^T,
\]

where \( \subset \) denotes strict inclusion. For vertices of the feasible payoff set, consider \( V = S_{1,1} \). The (interior) vertices of the Pareto frontier of \( V^T \) are given by

\[
Z^T(t) = \left( \frac{1 + \delta_I + \ldots + \delta_I^{t-1}}{1 + \delta_I + \ldots + \delta_I^{T-1}}, \frac{\delta_P^t + \ldots + \delta_P^{T-1}}{1 + \delta_P + \ldots + \delta_P^{T-1}} \right) \text{ for } t = 1, \ldots, T
\]
and the end points of the Pareto frontier are denoted as

$$Z^T (0) = (0, 1) \quad \text{and} \quad Z^T (T + 1) = (1, 0).$$

Finally, denote $Z^T (t) = (X^T (t), Y^T (t))$. Chen (2007) in addition shows that

$$X^{T+1} (t) < X^T (t) < X^{T+1} (t + 1), \quad \text{for} \quad t = 1, 2, \ldots, T,$$

and for each $t = 1, 2, \ldots, T$, vertex $Z^T (t)$ lies below the straight line going through points $Z^{T+1} (t)$ and $Z^{T+1} (t + 1)$. In particular, this implies that the Pareto frontier of the feasible payoff set expands strictly over time.\footnote{Hereafter, the Pareto frontier expands strictly over time if the Pareto frontier of $V^{T+1}$, $F^{T+1}$, is “above” the Pareto frontier of $V^T$, $F^T$ — that is, $F^{T+1}$ lies to the north-east of $F^T$ except for finitely many points.}

### 3.1 Characterization of the Vertices of $V^T$

We now characterize the vertices of $V^T$ for a general two-player finitely repeated game $G^T$. We only focus on the Pareto frontier of $V^T$ and analogous results can be obtained, \textit{mutatis mutandis}, for frontiers in other directions.

Suppose the initial (polyhedral) feasible payoff set $V$ is expressed as

$$V = \sum_{k=1}^{r} S_{p_k, q_k},$$

where $p_k, q_k \in \mathbb{R}_+$ for $k = 1, \ldots, r$, and $(r + 1)$ is the total number of vertices on the Pareto frontier $F$ (we also include the two end vertices). As $V$ is polyhedral, the Pareto frontier of $V$ is expressed as a translation of the Pareto frontier of $\sum_{k=1}^{r} S_{p_k, q_k}$ given by (7). To see the validity of the decomposition formula (7), notice that each edge of the Pareto frontier $F$ corresponds to a \textit{unique} triangle $S_{p, q}$ whose slope and slope length are the same as those of the edge on $F$.\footnote{For illustration, consider an initial feasible payoff set $V = co \{(0, 0), (0, 1), (1, 0), (p, q)\}$ where $p, q \in (0, 1)$ and $p + q > 1$. It can be verified that $V$ can be decomposed into the Minkowski sum of orthogonal triangles $S_{(1-p), q}$ and $S_{p,(1-q)}$.}

In this way, one can see that the decomposition (7) is \textit{unique}.\footnote{More precisely, we assume that we have distinct slopes $0 \leq q_1/p_1 < q_2/p_2 < \cdots < q_r/p_r$, where we allow $p_r = 0$ and regard $q_r/p_r = +\infty$.} Hereafter, we say that $V$ is \textit{degenerate} if $r = 1$ and “$q_1 = 0$ or $p_1 = 0$”, or if $r = 2$, and “$q_1 = 0$, $p_1 > 0$, $q_2 > 0$ and $p_2 = 0$.” Alternatively, the set $V$ is \textit{non-degenerate} if expression (7) contains at least one simplex $S_{p_k, q_k}$ with $p_k > 0$ and $q_k > 0$. Otherwise, $V$ is \textit{degenerate}.
It follows that

\[
V^T = \sum_{t=1}^{T} \left( \sum_{k=1}^{r} S_{p_k, q_k} \right) \left[ \gamma^T_I(t), \gamma^T_P(t) \right] 
= \sum_{t=1}^{T} \left( \sum_{k=1}^{r} S_{\gamma^T_I(t)p_k, \gamma^T_P(t)q_k} \right) 
= \sum_{k=1}^{r} \left( \sum_{t=1}^{T} S_{\gamma^T_I(t)p_k, \gamma^T_P(t)q_k} \right). 
\]

(8)

For any nonnegative weights \((\omega_1, \omega_2)\), consider the maximization problem:

**Problem \((P)\):** \[
\begin{align*}
\max_{(Z_1, Z_2) \in V^T} & \quad \omega_1 Z_1 + \omega_2 Z_2 \\
\text{subject to} & \quad (Z_1, Z_2) \in V^T.
\end{align*}
\]

By the definition of Minkowski sum and (8), Problem \((P)\) can be decomposed into \(rT\) separate maximization problems \((k = 1, \ldots, r\) and \(t = 1, \ldots, T)\):

**Problem \((P^{k,t})\):** \[
\begin{align*}
\max_{(Z_1, Z_2) \in V^T} & \quad \omega_1 Z_1 + \omega_2 Z_2 \\
\text{subject to} & \quad (Z_1, Z_2) \in S_{\gamma^T_I(t)p_k, \gamma^T_P(t)q_k}.
\end{align*}
\]

Now each Problem \((P^{k,t})\) over the triangle simplex \(S_{\gamma^T_I(t)p_k, \gamma^T_P(t)q_k}\) can be easily solved — in particular, in each \((P^{k,t})\), for generic values of \((\omega_1, \omega_2)\), the maximum of \((\omega_1 Z_1 + \omega_2 Z_2)\) over \(S_{\gamma^T_I(t)p_k, \gamma^T_P(t)q_k}\) is obtained on one of the two vertices \((\gamma^T_I(t)p_k, 0)\) and \((0, \gamma^T_P(t)q_k)\). Denoting the optimal solution of Problem \((P^{k,t})\) by \((Z^{k,t}_1, Z^{k,t}_2)\), we then obtain an optimal solution \((Z^*_1, Z^*_2)\) of Problem \((P)\) given by

\[
(Z^*_1, Z^*_2) = \sum_{k=1}^{r} \sum_{t=1}^{T} \left( Z^{k,t}_1, Z^{k,t}_2 \right) = \sum_{t=1}^{T} \sum_{k=1}^{r} \left( Z^{k,t}_1, Z^{k,t}_2 \right). 
\]

(9)

Observe that the last term of (9) enables us to obtain the specific action profile of the stage game in each time instance \(t\) that yields \((Z^*_1, Z^*_2)\).

If we consider the weights \((\omega_1, \omega_2)\) parametrically such as \((\omega, 1 - \omega)\) with \(0 \leq \omega \leq 1\), the optimal solution \((Z^{k,t}_1, Z^{k,t}_2)\) of Problem \((P^{k,t})\) jumps from \((0, \gamma^T_P(t)q_k)\) to \((\gamma^T_I(t)p_k, 0)\) as \(\omega\) increases from 0 to 1. The jumping value of \(\omega\) is equal to \(\frac{\gamma^T_P(t)q_k}{\gamma^T_I(t)p_k + \gamma^T_P(t)q_k}\).

This fact, together with (9), characterizes the vertices of \(V^T\): Each vertex of \(V^T\) is the sum of the vertices of \(S_{\gamma^T_I(t)p_k, \gamma^T_P(t)q_k}\), each being given as an optimal solution of Problem \((P^{k,t})\) for a common pair of weights, for \(k = 1, \ldots, r\) and \(t = 1, \ldots, T\).
3.2 A Comparative Statics Result for $V^T$

We now present a comparative statics result, analogous to that in Chen (2007): Fixing the players’ discount factors, let the length of the time horizon vary. As the time horizon increases, the Pareto frontier of the feasible payoff set of $G^T$ expands strictly over time (see Footnote 4), and the feasible set increases in the set-inclusion sense.

**Proposition 1** Consider a two-player repeated game $G^T$ with unequal discounting where at least one of the frontiers of the polyhedral payoff set $V$ is non-degenerate. We have

1. The Pareto frontier expands strictly over time;
2. A strict inclusion result for the feasible set:

\[ V^1 \subset V^2 \subset \cdots \subset V^T. \]

**Proof.** Suppose that the initial feasible set $V$ is decomposed as in (7). As shown before, we can then express $V^T$ as

\[ V^T = \sum_{k=1}^{r} \left( \sum_{t=1}^{T} S_{\gamma(t)p_k,\gamma(t)q_k} \right). \]

This expression, together with the strict monotonicity result (6) in Chen (2007), implies that

\[ V^1 \subset V^2 \subset \cdots \subset V^T, \tag{10} \]

where notice that if $U_1 \subset U_2 \subseteq \mathbb{R}^2$ and $W_1 \subset W_2 \subseteq \mathbb{R}^2$, then we have $U_1 + W_1 \subset U_2 + W_2$ for Minkowski sums. In particular, if we allow degenerate triangles $S_{p,0}$ and $S_{0,q}$ for $p, q > 0$, then it is possible that all $V^t, t = 1, 2, \ldots, T$, will be the same (for example, consider $V = S_{p,0} + S_{0,q}$). More precisely, the strict monotonicity result (10) holds true if and only if at least one summand of $V$ in (7) is not degenerate.

Next, suppose that the initial feasible set $V$ is expressed as

\[ V = \omega_0 + \sum_{k=1}^{r} S_{p_k,q_k}. \]
for an arbitrary vector $\omega_0 \in \mathbb{R}^2$ and positive real numbers $p_k$ and $q_k$ $(k = 1, \ldots, r)$.

Then we have

$$V^T = \sum_{t=1}^{T} \left( \omega_0 + \sum_{k=1}^{r} S_{p_k,q_k} \right) \left[ \gamma^T_I (t), \gamma^T_P (t) \right]$$

$$= \sum_{t=1}^{T} \left( \omega_0 \left[ \gamma^T_I (t), \gamma^T_P (t) \right] + \sum_{k=1}^{r} S_{p_k,q_k} \left[ \gamma^T_I (t), \gamma^T_P (t) \right] \right)$$

$$= \omega_0 + \sum_{k=1}^{r} \left( \sum_{t=1}^{T} S_{\gamma^T_I (t)p_k, \gamma^T_P (t)q_k} \right).$$

Because of (10), the above implies that for any polyhedral initial feasible set, the positive (Pareto) frontier shifts to the north-east direction (strictly) as the time horizon increases.

We can further apply the above argument, *mutatis mutandis*, to frontiers in other directions.

In summary, given any polyhedral initial feasible set $V$, we have

$$V^1 \subset V^2 \subset \cdots \subset V^T,$$

except for the degenerate case (in all directions) mentioned above. ■

**Further Remarks:**

(1) The Degenerate Case: Proposition 1 does not hold when all summands of $V$ in (7) are degenerate triangles, in the sense that we describe in the paragraph after (7). Figure 1 provides such an example:

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<thead>
<tr>
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<th>$L$</th>
<th>$R$</th>
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</thead>
<tbody>
<tr>
<td>$U$</td>
<td>(1, 1)</td>
<td>(-1, 1)</td>
</tr>
<tr>
<td>$D$</td>
<td>(1, -1)</td>
<td>(-1, -1)</td>
</tr>
</tbody>
</table>

Figure 1. The stage game $\hat{G}$ of a repeated game $G^2$ where Proposition 1 fails.

In this example, there is no gain from trading intertemporally for all $(\delta_I, \delta_P) \in (0, 1)^2$. Such games are the *only* games where intertemporal trade has no bite in a two-player repeated game with unequal discounting.

(2) Public Randomization: Throughout the paper, we have assumed the existence of a public randomization device, which enables the players to play any correlated action profile in each period. While this assumption is in line with the previous literature on repeated games, Proposition 1 is no longer true without public randomization. Indeed, in the absence of public randomization, the feasible payoff set of $G^T$ is highly sensitive to the discount factors $(\delta_I, \delta_P)$. As a result, the strict monotonicity

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7This step could be replaced by properly normalizing the players' stage payoffs while maintaining the strategic aspects of $G$.  

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result in Proposition 1 which is independent of \((\delta_I, \delta_P)\) (except that \(1 > \delta_P > \delta_I > 0\)) would in general not hold.⁸

(3) n-player Repeated Games: While a result like Proposition 1 seems natural and intuitive for n-player repeated games with unequal discounting as well, our arguments in the paper depend heavily on the dimensionality of two. In particular, the crucial decomposition formula (7) is in general possible only for the two-dimensional case, implying that a similar argument extending the result to a higher dimensional case is difficult.

References


⁸Salonen and Vartiainen (2008) show that the feasible set of certain two-player infinitely repeated games with unequal discounting may be totally disconnected (i.e., all its components are single point sets). The indispensability of public randomization has also been demonstrated in several other studies on repeated games. Olszewski (1998) proves that folk theorems need not hold in some two-player repeated games where one player can use only pure strategies. Yamamoto (2010) shows via an example that the set of subgame-perfect equilibrium payoffs of the repeated game is not convex and the set of pure-strategy equilibrium payoffs is not monotonic in the discount factor.