On Effective Minimax Payoffs and Unequal Discounting

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Abstract

We show that the Folk theorem in Wen (1994) may not fully characterize the subgame-perfect equilibrium payoff set in a repeated game with unequal discounting, where a player’s equilibrium payoff could be strictly less than her effective minimax payoff.

Keywords: repeated games, effective minimax values, heterogenous discounting

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1 Introduction

In the study of repeated games, one central theme is to characterize the set of equilibrium payoffs, or folk theorems. In a seminal paper, Fudenberg and Maskin (1994) establish the classical folk theorem which states that under certain conditions, any feasible and individually rational payoff of the stage game is a (subgame perfect) equilibrium payoff in the corresponding repeated game. The classical folk theorem requires that the set of payoff vectors have "full dimension" for repeated games with more than two players. This full dimensionality condition is weakened and replaced by the "non-equivalent utilities" (NEU) condition in Abreu, Dutta and Smith (1994), which states that no two players have identical preferences over action profiles in

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the stage game. For repeated games that fail the NEU condition, Wen (1994) introduces the concept of effective minimax values and characterizes the set of equilibrium payoffs with little discounting.

In this paper, I consider a simple three-player infinitely repeated game where the NEU condition fails. I show that if players have different discount factors, some player may obtain a subgame perfect equilibrium payoff that is strictly lower than her effective minimax value.\footnote{One reason why we should worry about low equilibrium payoffs is that lower equilibrium payoffs can serve as credible punishments to provide incentives for the players to cooperate on other equilibrium paths.} I also show that this is the case no matter how patient the player is.

This paper is also related to the literature on repeated games with unequal discounting. Lehrer and Pauzner (1999) first formally study the effect of introducing unequal discounting on repeated games. They show that the feasible payoff set of a general two-player infinitely repeated game is larger than the convex hull of the stage-game payoffs, as a result of players’ being able to mutually benefit from trading payoffs across time. Lehrer and Pauzner also characterize the set of equilibrium payoffs in two-player infinitely repeated games and they find that, in contrast to the classical folk theorem in Fudenberg and Maskin (1986), not all feasible individually rational payoffs can be supported by an equilibrium, even when both players become very patient.

The results in Lehrer and Pauzner (1999) are mainly about repeated games with two players. For more general $n$-player repeated games, Lehrer and Pauzner point out that one complication in obtaining a folk theorem result in these games is that it is difficult to characterize the feasible payoff set of a general $n$-player repeated game with unequal discounting. In this paper, we point out yet another complication in characterizing the equilibrium payoffs for a general $n$-player repeated game with unequal discounting: the lower bound of equilibrium payoffs in such repeated games crucially depend on the players’ discount factors when there is unequal discounting.

\section{The Example}

Consider the following three-player infinitely repeated games with perfect monitoring. The stage game is defined below in Figure 1, where player 1 chooses rows, player 2
columns, and player 3 matrices.

\[
\begin{array}{ccc}
T & L & R \\
-1, 2, -1 & 0, 0, 0 & 0, 0, 0 \\
M & 0, 0, 0 & 0, 1, 0 & 0, 1, 0 \\
B & -2, 2, -2 & -1, 0, -1 & -1, 0, -1 \\
C & D
\end{array}
\]

Figure 1: The Stage Game of a Three-Player Infinitely-Repeated Game with Unequal Discounting \((\delta_1 > \delta_3)\).

In this repeated game, the full dimensionality condition of Fudenberg and Maskin (1986) and the non-equivalent utilities (NEU) condition of Abreu, Dutta and Smith (1994) are violated as player 1 and player 3 have equivalent utilities, or \(I = \{1, 3\}\), where \(I\) is the set of players who have equivalent utilities. The effective minimax value of player 1 and player 3 is \(v^e_i = v^e_j = 0\), and player 3’s (individual) minimax value is \(-1\).\(^2\) Suppose players have discount factors as \(1 > \delta_1 > \delta_3 > 0\). We now construct a pure-strategy subgame perfect equilibrium where player 3 obtains a subgame perfect equilibrium payoff that can be arbitrarily close to \(-1\), as long as player 1 is sufficiently more patient than player 3.

- On the equilibrium path, players play \((B, L, C)\) for \(N\) periods (\(N\) to be defined), after which, \((T, L, C)\) is played for the rest of the game;
- In the first \(N\) periods, if player 1 deviates, then players play \((M, R, C)\) for the rest of the game; any deviation in the first \(N\) periods made by either player 2 or player 3 is ignored.

As \((T, L, C)\) and \((M, R, C)\) are two stage game Nash equilibria, it suffices to verify that player 1’s incentives are correct in any of the first \(N\) periods on the equilibrium path. As player 1 discounts future payoffs, a sufficient condition is that player 1 would not deviate in the first period:

\[
(1 - \delta_1^N) (-1) + \delta_1^N = 2\delta_1^N - 1 > (1 - \delta_1),
\]

which is satisfied as long as \(\delta_1\) is large enough.

\(^2\)The effective minimax value (Wen (1994)) for player \(i\), where \(i \in I\), is defined as:

\[
\text{v}^e_i = \min_a \max_{j \in I} u_i(a_j, a_{-j}),
\]

where \(a\) is an (observable) mixed action profile and \(u_i(\cdot)\) denotes player \(i\)’s payoff in the stage game. In other words, player \(i\)’s (\(i \in I\)) effective minimax value is defined as player \(i\)’s minimum payoff under the best unilateral deviation by some player \(j \in I\) (but not a joint deviation by players in \(I\)). The above effective minimax value is, however, defined under the assumption that mixed actions are observable. Fudenberg, Levine and Takahashi (2007) modifies Wen’s effective minimax values and characterizes the set of equilibrium payoffs when mixing probabilities are not observable.
Player 3’s equilibrium payoff is \( (2\delta_3^N - 1) \), which can be made very close to \(-1\), if \( \delta_3 \) is sufficiently small and/or \( N \) is large enough. Recall that this also requires \( \delta_1 \) to be large enough so player 1’s incentives are correct.

Specifically, one can show that for any \( \delta_3 \in (0, 1) \) and any \( \epsilon \in (0, 1) \), there exist some \( \delta_1 \in (\delta_3, 1) \) and some \( N \in \mathbb{N} \), such that the following is true:

\[
\begin{align*}
2\delta_1^N + \delta_1 &> 2 \\
2\delta_3^N &< \epsilon
\end{align*}
\]

The reason is the following: given any \( \delta_3 \in (0, 1) \) and any \( \epsilon \in (0, 1) \), define an \( N \in \mathbb{N} \) such that \( N > \frac{\ln(\epsilon/2)}{\ln \delta_3} \), which is some real number in \((0, +\infty)\); furthermore, notice that \( (2\delta_1^N + \delta_1) \) is increasing in \( \delta_1 \) for any fixed \( N \) and \( \lim_{\delta_1 \to 1} (2\delta_1^N + \delta_1) = 3 \). Hence the result.

The above argument shows that no matter how patient player 3 is, there exists a subgame perfect equilibrium of the infinitely repeated game where player 3’s equilibrium payoff can be arbitrarily close to her individual minimax value, which is strictly lower than her effective minimax value 0, as long as player 1 is sufficiently more patient than player 3.

To build more content into the above result, consider the following scenario behind the small example. Suppose three players repeatedly undertake a profitable project and \( \delta_1 > \delta_2 \) and \( \delta_1 \) is sufficiently greater than \( \delta_3 \). Player 1 and player 2, who hold different time preferences over future payoffs, plan some "conspiracy" against player 3: as \( \delta_1 > \delta_2 \), player 1 and player 2 conduct certain intertemporal trade (in the sense of Lehrer and Pauzner (1999)) by first allowing player 2 to obtain a large share from the project, and then letting player 1 obtain large profits afterwards. This intertemporal trade, while most likely benefiting both player 1 and player 2, hurts player 3 as relatively she cares more about short-run payoffs. However, player 3 can do nothing about it as she has little control over the project: player 3’s payoff in each stage is mainly determined by player 1 and player 2’s actions as her own actions have little effect on changing her own, as well as others’, payoffs in each stage.

The three-player example, where some player obtains a payoff strictly lower than her effective minimax value in an equilibrium, is not peculiar for \( n \)-player repeated games where the NEU condition fails, since afterall, the very idea of effective minimax values hinges on the fact that some player (call it player \( i \)), who has little control over the worst scenarios of the repeated game, "free rides" on the actions of some other player (call it player \( j \)) in \( I \), the set of players who have equivalent utilities, in the game. In other words, player \( i \), who is vulnerable under worst scenarios, is "protected" by player \( j \), who has a better control in the bad situations, as player \( i \) and player \( j \) share common interest in the game. As the game is repeated, such "free-ride" behavior is not affected when players in \( I \) have a common discount factor. However, if players in \( I \) view future payoffs differently, such "free-riding" may no longer be in place when the game is repeated. As a result, player \( i \) may obtain a
lower equilibrium payoff than her effective minimax value in the repeated game. This is also the intuition of the result of our simple example.

3 Conclusion

In this note, we present a simple example of a three-player infinitely repeated game where the NEU condition is failed and players have unequal discounting. We show that some player may obtain a subgame perfect equilibrium payoff that is strictly lower than her effective minimax value. Our example thus implies that there is some non-trivial relationship between effective minimax values and players’ discount factors. Imaginably, a more appropriate effective minimax value for a repeated game with unequal discounting should crucially depend on the discount factors of the players in $I$, as well as the structure of the stage game, in the same vein as the definition of minimax values in Dutta (1995) for stochastic games. On the other hand, a reasonable conjecture is that as long as the stage game satisfies the non-equivalent utility (NEU) condition of Abreu, Dutta and Smith (1994), players’ lower bounds of equilibrium payoffs would not be affected by unequal discounting.

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References


