Networks with Side Constraints: An LU Factorization Update

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An important class of mathematical programming models which are frequently used in logistics studies is the model of a network problem having additional linear constraints. A specialization of the primal simplex algorithm which exploits the network structure can be applied to this problem class. This specialization maintains the basis as a rooted spanning tree and a general matrix called the working basis. This paper presents the algorithms which may be used to maintain the inverse of this working basis as an LU factorization, which is the industry standard for general linear programming software. Our specialized code exploits not only the network structure but also the sparsity characteristics of the working basis. Computational experimentation indicates that our LU implementation results in a 50 percent savings in the non-zero elements in the eta file, and our computer codes are approximately twice as fast as MINOS and XMP on a set of randomly generated multicommodity network flow problems.

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Good software for solving linear programming models is one of the most important tools available to the logistics engineer. For logistics studies, these linear programs frequently involve a very large network of nodes and arcs, which may be duplicated by time period. For example, nodes may represent given cities at a particular point in time while arcs represent roads, railways, and legs of flights connecting these cities. Some nodes are designated as supply nodes, others demand nodes, while some may simply represent points of transshipment. The mathematical model characterizes a solution such that the supply is shipped to the demand nodes at least cost while not violating either the upper or lower bounds on the flow over an arc.

If the major structure of a logistics problem can be captured in a network model, then the size of solvable problems becomes enormous. Hence, more realistic situations can be modeled that would otherwise lie outside the domain of general linear programming techniques. For example, one current logistics planning model involves 200 nodes and (365 days/yr) (30 years) = 10,950 time periods to give over 2,000,000 constraints. Network problems having 20,000 constraints and 20,000,000 variables are solved routinely at the U.S. Treasury Department.

Unfortunately, the pure network structure may require simplification of the problem to the point that key policy restrictions must be omitted. The work presented in this study builds upon existing large-scale network solution technology to allow for the inclusion of arbitrary additional constraints. Typical constraints include capacities on vehicles carrying different types of goods, restrictions on the total number of vehicles available for assignment, and budget restrictions. The addition of even a few non-network constraints can greatly enhance the realism and usability of these models. Our approach exploits—to as great an extent as possible—the traditional network portion of the problem while simultaneously enforcing any additional restrictions imposed by the practitioner.

For general linear programming systems, the most important component is the algorithm used to update the basis inverse. Due to the excellent sparsity and numerical stability characteristics, an LU factorization with either a Bartels-Golub or Forrest-Tomlin update has been adopted for modern linear programming systems. For pure network problems, the basis is always triangular and corresponds to a rooted spanning tree. The modern network codes which exploit this structure have been found to be from one to two orders of magnitude faster than the general linear programming systems. In this paper, we have combined these two powerful techniques into an algorithm for solving network models having additional side constraints.

Let \( A \) be an \( m \times n \) matrix, let \( c \) and \( u \) be \( n \)-component vectors, and let \( b \) be an \( m \)-component vector. Without loss of generality, the linear program may be stated mathematically as follows:
minimize \[ c^T x \] 
subject to: \[ A^T y = b \] 
\[ 0 \leq x \leq u. \]

The network with side constraint model is a special case of (1)–(3) in which \( A \) takes the form:

\[ A = \begin{bmatrix} M & S \end{bmatrix} \in \mathbb{R}^{m \times n} \]

where \( M \) is a node-arc incidence matrix.

If \( m = 0 \), then (1) – (3) is a pure network problem.

1.1 Applications

There are numerous applications of the network with side constraint model. Professor Glover and his colleagues have solved a large passenger-mix model for Frontier Airlines and a large land management model for the Bureau of Land Management (see [7, 8]). A world grain export model has been solved to help analyze the port capacity of U.S. ports during the next decade (see [2]). A cargo routing model is being used by the Air Force Logistics Command to assist in routing cargo planes for the distribution of serviceable spares (see [1]). Lt. Col. Dennis McLain, has developed a large model to assist in the development of a casualty evacuation plan in the event of a European conflict (see [14]). A National Forest Management Model has been developed to aid forest managers in long term planning for national forests (see [10]). In addition, work is currently underway which attempts to convert general linear programs into the network with side constraint model (see [4, 16]).

1.2 Objective of Investigation

Due to both storage and time considerations, the basis inverse is maintained as an LU factorization in modern LP software (see [3, 5, 15]). The objective of this investigation is to extend these ideas to the primal partitioning algorithm when applied to the network with side constraints model.

1.3 Notation

The \( i \)th component of the vector \( a \) will be denoted by \( a_i \). The \( (i, j) \)th element of the matrix \( A \) is denoted by \( A_{ij} \). \( A(i) \) and \( A(j) \) denote the \( i \)th column and \( j \)th row of the matrix \( A \), respectively. \( 0 \) denotes a vector of zeroes, \( 1 \) denotes a vector of ones, and \( e_k \) denotes a vector with a 1 in the \( k \)th position and zeroes elsewhere. \( \sigma \) is used to denote the scalar signum function defined by
\[ \sigma(y) = \begin{cases} 
1, & \text{if } y > 0 \\
0, & \text{if } y = 0 \\
-1, & \text{if } y < 0.
\end{cases} \]

The identity matrix is given by "I".

II. THE PRIMAL SIMPLEX ALGORITHM

We assume that \( A \) has full row rank and that there exist a feasible solution for (1)–(3). Given a basic feasible solution, we may partition \( A, c, x, \) and \( u \) into basic and nonbasic components, that is, \( A = [B|N] \), \( c = [c^B|c^N] \), \( x = [x^B|x^N] \), and \( u = [u^B|u^N] \). Using the above partitioning, the primal simplex algorithm may be stated as follows:

**PRIMAL SIMPLEX ALGORITHM**

1. **Initialization.** Let \( (x^0|x^N) \) be a basic feasible solution.

2. **Pivot.** Let \( \pi = c^B B^{-1} \). Define
   \[ \psi_1 = \{i \mid x_i^N = 0 \text{ and } \pi N(i) > c_i^N\}, \]
   \[ \psi_2 = \{i \mid x_i^N = u_i^N \text{ and } \pi N(i) < c_i^N\}. \]
   If \( \psi_1 \cup \psi_2 = \emptyset \), terminate with \( (x^0|x^N) \) optimal; otherwise, select \( k \in \psi_1 \cup \psi_2 \), and set \( \delta \leftarrow 1 \) if \( k \in \psi_1 \), and \( \delta \leftarrow -1 \), otherwise.

3. **Ratio Test.** Set \( y \leftarrow B^{-1} N(k) \). Set
   \[ \Delta_1 \leftarrow \sigma(y) \leftarrow \frac{x_i^B}{y_i}, \quad \Delta_2 \leftarrow \sigma(y) \leftarrow \frac{x_i^0 - x_i^B}{|y_i|}. \]
   Set \( \Delta \leftarrow \min(\Delta_1, \Delta_2, u_i^N) \).
   If \( \Delta \neq \infty \), then go to 3; otherwise, terminate with the conclusion that the problem is unbounded.

4. **Update Values.** Set \( x_i^N \leftarrow x_i^N + \Delta \delta \) and \( x_i^B \leftarrow x_i^B - \Delta \delta y \). If \( \Delta = u_i^N \), return to step 1.

4. **Update Basis Inverse.** Let
   \[ \psi_0 = \{i \mid x_i^B = 0 \text{ and } \sigma(y) = \delta\}, \]
   \[ \psi_1 = \{i \mid x_i^B = u_i^B \text{ and } -\sigma(y) = \delta\}. \]
   Select any \( i \in \psi_0 \cup \psi_1 \). In the basis, replace \( B(t) \) with \( N(k) \), update the inverse of the new basis, and return to step 1.
III. THE PARTITIONED BASIS

The network with side constraint model may be stated as follows:

\[
\text{minimize} \quad c^1 x^1 + c^2 x^2
\]

subject to:
\[
M x^1 = b^1
\]
\[
S x^1 + P x^2 = b^2
\]
\[
0 \leq x^1 \leq u^1
\]
\[
0 \leq x^2 \leq u^2
\]

(4) (5) (6) (7) (8)

We may assume without loss of generality that,

(i) The graph associated with \( \mathcal{M} \) has \( n \) nodes and is connected (i.e., there exists an undirected path between every pair of nodes).
(ii) \([S;P]\) has full row rank (i.e., rank \([S;P]\) = \( m \)).
(iii) Total supply equals total demand (i.e., \( 1 b^* = 0 \)).

Since the rank of system (5) is one less than the number of rows, we add what has been called the root arc to (5) to obtain

\[
M x^1 + e^p a = b^1
\]

where \( 0 \leq a \leq 0 \) and \( 1 \leq p \leq n \).

Then the constraint matrix for the network with side constraints model becomes

\[
A = \begin{bmatrix} M & e^p \\ S & P \end{bmatrix}
\]

and \( A \) has full row rank.

It is well-known that every basis for \( A \) may be placed in the form

\[
B = \begin{bmatrix} T & C \\ D & F \end{bmatrix}
\]

(9)

where \( T \) corresponds to a rooted spanning tree and

\[
B^{-1} = \begin{bmatrix} T^{-1} + T^{-1} C Q^{-1} D T^{-1} & -T^{-1} C Q^{-1} \\ -Q^{-1} D T^{-1} & Q^{-1} \end{bmatrix}
\]

(10)

where \( Q = F - D T^{-1} C \). The objective of this paper is to give algorithms which maintain \( Q^{-1} \) as an LU factorization.

IV. THE INVERSE UPDATE

Recall that the partitioned basis takes the form

\[
B = \begin{bmatrix} T & C \\ D & F \end{bmatrix}
\]

key
nonkey
Let
\[ L = \begin{bmatrix} T^{-1} & -T^{-1}C \\ \ast & I \end{bmatrix} \]
and let
\[ \bar{B} = BL = \begin{bmatrix} I & T^{-1}C \\ DT^{-1} & Q \end{bmatrix}. \]

The inverse update requires a technique for obtaining a new \( Q^{-1} \) after a basis exchange. Let \( \bar{B}, L, B, \) and \( Q \) denote the above matrices at iteration \( i \). Then we want an expression for \( Q_{i+1}^{-1} \) in terms of \( Q_i^{-1} \). The transformation takes the form
\[ B_{i+1}^{-1} = \bar{E} B_i^{-1} \]
(11)
where \( \bar{E} \) is either an elementary column matrix or a permutation matrix. Let \( \bar{E} \) be partitioned to be compatible with \( B \). That is,
\[ \bar{E} = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix} \]
where
\[ E = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix} \]
and
\[ Q_{i+1}^{-1} = (E_4 - E_3T^{-1}C)Q_i^{-1} \]
(12)

By examining the (2,2) partition of \( B_{i+1}^{-1} \), we obtain
\[ Q_{i+1}^{-1} = (E_4 - E_3T^{-1}C)Q_i^{-1} \]
(12)

In determining the updating formulae, we must examine two major cases with subcases.

**Case 1.** The leaving column is nonkey. For this case, \( \bar{E} \) takes the form
\[ \begin{bmatrix} 1 & E_2 \\ \ast & E_3 \end{bmatrix} \]
and (12) reduces to \( Q_{i+1}^{-1} = E_4Q_i^{-1} \).

**Case 2.** The leaving column is key.

Let \( \gamma = e^T \gamma \). If \( \gamma_k \neq 0 \), then the \( k \)th column of \( C \) can be interchanged with the \( j \)th column of \( T \) and the new \( T_j \) will be nonsingular.

**Subcase 2a.** \( \gamma 
eq 0 \). Suppose \( \gamma_k \neq 0 \). Then
\[ E_4 - E_3T^{-1}C \]
reduces to
\[ R = \begin{bmatrix} 1 & -e^T \gamma \\ \ast & I \end{bmatrix} \]
(13)
\[ \text{row } j \]
and
\[ Q_{i+1}^{-1} = RQ_i^{-1} \]. Case 1 is applied to complete the update.

**Subcase 2b.** \( \gamma = 0 \). For this case no interchange is possible, the entering column becomes key, and \( Q_{i+1}^{-1} = Q_i^{-1} \).
V. AN LU UPDATE

Let

\[ U^l = \begin{bmatrix} \mu_1 & \epsilon_1 & 0 \\ \vdots & \vdots & \vdots \\ \mu_{m-1} & \epsilon_{m-1} & 0 \\ 0 & \cdots & 1 \end{bmatrix} \]

and

\[ L^l = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \epsilon_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \epsilon_m \end{bmatrix} \]

Matrices of the form given by \( U^l \) and \( L^l \) are called upper etas and lower etas, respectively. Suppose we have a factorization of \( Q^{-1} \) in the form

\[ Q^{-1} = U^l U^2 \cdots U^n F^s \cdots F^i \cdots F^1, \quad (14) \]

where \( F^1, \ldots, F^s \) are a combination of row and column etas. The right side of (14) is referred to as the eta file where only the non-identity rows and columns are stored. Suppose that the \( k^{th} \) column of \( Q \) is replaced by \( \hat{Q}(k) \) to form the new m by m working basis \( \hat{Q} \). This section presents algorithms which may be used to update (14) to produce \( \hat{Q}^{-1} \) in the same form.

5.1 Nonkey Column Leaves The Basis

If \( k = m \), then let \( \beta = F^m \cdots F^1 \hat{Q}(k) \), let

\[ \hat{U}^m = \begin{bmatrix} 1 & \vdots & 1/eta_m \end{bmatrix} \]

and let

\[ \hat{U}^m = \begin{bmatrix} 1 & -\beta_1 & \vdots & -\beta_m \end{bmatrix} \]
We will show that $\hat{Q}^{-1} = U^1 \ldots U^{m-1}\hat{U}\ldots F^1$.

If $k < m$, then let $R^k = I$ and

$$Q^{-1} = U^1 \ldots U^kR^kU^{k+1} \ldots U^mF^k \ldots F^1.$$  \hspace{1cm} (15)

We next define a new upper eta, $\hat{U}^k$, and a new row eta, $R^{k+1}$, such that

$$R^kU^{k+1} = \hat{U}^kR^{k+1}.$$  \hspace{1cm} (16)

Substituting (16) into (15) yields

$$Q^{-1} = U^1 \ldots U^k\hat{U}^{k+1} \ldots U^mF^k \ldots F^1.$$  \hspace{1cm} (17)

We again define two new eta’s, $\hat{U}^{k+1}$ and $R^{k+2}$, such that

$$R^{k+1}U^{k+2} = \hat{U}^{k+1}R^{k+2}.$$  \hspace{1cm} (18)

Substituting (18) into (17) yields

$$Q^{-1} = U^1 \ldots U^k\hat{U}^{k+1} \ldots U^{m-1}\hat{U}^{m}F^k \ldots F^1.$$  \hspace{1cm} (19)

Let $\gamma = R^mF^k \ldots F^1\hat{U}(k)$, let

$$\hat{L}^m = \begin{bmatrix}
1/\gamma_k \\
-\gamma_{k-1}/\gamma_k \\
\vdots \\
-\gamma_1/\gamma_k
\end{bmatrix}
$$

and let

$$\hat{U}^m = \begin{bmatrix}
-\gamma_1 \\
\vdots \\
-\gamma_k \\
1
\end{bmatrix}
$$

Then, $\hat{U}^m\hat{L}^m\gamma = e^k$ and we will show that $\hat{Q}^{-1} = U^1 \ldots U^{k-1}\hat{U} \ldots \hat{U}^m\hat{L}^m\gamma \ldots F^1$.

We now present the algorithm which updates the LU representation of $Q^{-1}$ when the leaving column is nonkey. Assume that $\hat{U}(k)$ is replacing $Q(k)$ in the working basis.
**ALG 1: LU UPDATE FOR NONKEY LEAVING COLUMN**

1. Set \( \beta \leftarrow F^k \ldots F^1 Q(k) \).
2. If \( k \leq m \), set \( \ell \leftarrow k \), \( R^\ell \leftarrow I \), go to 4.
3. Set \( U^i \leftarrow I \), where \( i \) is \( m \) by \( m \).
   
   Set \( U_{m,m}^i \leftarrow 1/\beta_m \).
   
   Set \( U_{i,j}^i \leftarrow 0 \), where \( 1 \leq j < m \).
   
   Stop with \( Q^{-1} = U^1 \ldots U^{m-1} U^m \).
4. Set \( \alpha \leftarrow R^1(k)U^{m+1}(\ell+1) \).
   
   Set \( R^\ell + 1 \leftarrow R^\ell \).
   
   Set \( R_{k+1}^\ell \leftarrow \alpha \).
   
   Set \( U_{(k+1),k}^\ell \leftarrow U_{k+1}^\ell \).
   
   Set \( U_{k+1,k}^\ell \leftarrow 0 \).
   
   \( (R^\ell U^{k+1} = U^{(R^\ell + 1)} \).
   
   Set \( \ell \leftarrow \ell + 1 \).
5. If \( \ell < m \), go to 4.

\[(U^{k+1} \ldots U^m = \bar{U}^{k} \ldots \bar{U}^{m-1} R^m)\]

Set \( \beta \leftarrow R^m \beta \).

6. Set \( U^m \leftarrow I \), where \( i \) is \( m \) by \( m \).
   
   Set \( U_{m,m}^i \leftarrow 1/\beta_m \).
   
   Set \( U_{i,j}^i \leftarrow -\beta_i/\beta_j \), for \( k < j \leq m \).
   
   Set \( U^i \leftarrow i \), where \( i \) is \( m \) by \( m \).
   
   Set \( U_{i,j}^i \leftarrow -\beta_i \), for \( 1 \leq j = k \).
   
   Set \( U_{m,i}^i \leftarrow 1 \).
   
   Stop with \( Q^{-1} = U^1 \ldots U^{m-1} U^m \).

We now present the justification for step 3 of ALG 1. For \( k = m \), we claim that \( Q^{-1} = U^1 \ldots U^{m-1} U^m \). Note that \( Q^{-1} Q(m) = U^1 \ldots U^{m-1} U^m \beta \). But by construction \( U^m \beta = \epsilon^m \). Consider

**Proposition 1.**

Let \( \beta \) be any \( m \)-vector and \( E^i \) be any column; \( \epsilon \). If \( \beta = 0 \), then \( E^i \beta = \beta \).

By Proposition 1, \( U^1 \ldots U^{m-1} \epsilon^m = \epsilon^m \). Therefore, \( Q^{-1} \epsilon^m = \epsilon^m \).

For \( 1 \leq k < m \), let \( \gamma = F^k \ldots F^1 Q(k) \). By construction \( \gamma_i = 0 \) for \( k < j \leq m \) and \( \gamma_k = 1 \). By Proposition 1, \( U^k+1 \ldots U^{m-1} U^m \gamma = \gamma \). By the construction of \( U^1 \ldots U^k \), we have \( U^1 \ldots U^k \gamma = \epsilon^k \). Therefore, if the leaving column is \( Q(m) \), then step 3 of ALG 1 produces \( Q^{-1} \).

We now present a theoretical justification for step 4 of ALG 1.
Proposition 2.

Let

\[ U^{p+1} = \begin{bmatrix} | & | & \cdots & | \\ \hline & \eta & & 1 \\ \hline \end{bmatrix} \quad \text{and} \quad R^p = \begin{bmatrix} | & \cdots & | \\ \hline & \gamma & \cdots & 1 \\ \hline \end{bmatrix} \]

where \( \ell \neq \ell^* \).

If

\[ \tilde{U}^p = \begin{bmatrix} | & | & \cdots & | \\ \hline & \alpha & & 1 \\ \hline \end{bmatrix} \quad \text{and} \quad R^{p+1} = \begin{bmatrix} | & \cdots & | \\ \hline & \beta & \cdots & 1 \\ \hline \end{bmatrix} \]

where

\[ \alpha_i = \begin{cases} 0, & \text{if } i = \ell^* \\ \eta_i, & \text{otherwise, and} \end{cases} \]

\[ \beta_i = \begin{cases} \eta_i \gamma, & \text{if } i = \ell \\ \gamma_i, & \text{otherwise,} \end{cases} \]

then \( R^p U^{p+1} = \tilde{U}^p R^{p+1} \).

Proposition 2 is a theoretical justification for step 4 of ALG 1. The proposition to follow shows the precise structure of \( R^m F^3 \ldots F^1 \). Consider

Proposition 3.

Let \( U^* = F^3 \ldots F^1 \). If \( \tilde{U}^* = R^n U^* \), then

\[ \tilde{U}^*ij = \begin{cases} U^*[i], & \text{if } i \neq k \\ e^k, & \text{otherwise.} \end{cases} \]

We now present the results to prove that \( \tilde{Q}^{-1} = U^1 \ldots U^{k-1} \tilde{U}^k \ldots \tilde{U}^* \tilde{U} \tilde{U}^{m+1} R^n F^3 \ldots F^1 \).

Proposition 4.

\[ U^1 \ldots U^{k-1} \tilde{U}^k \ldots \tilde{U}^{m+1} R^n F^3 \ldots F^1 (k) = e^k \]

Proposition 5.

\[ U^1 \ldots U^{k-1} \tilde{U}^k \ldots \tilde{U}^{m+1} R^n F^3 \ldots F^1 (i) = e^i \quad \text{for } i \neq k. \]
By Propositions 4 and 5, we have

$$\tilde{Q}^{-1} = U^1 \ldots U^{k-1} \tilde{U}^k \ldots \tilde{U}^m \tilde{R}^{m+1} \ldots F^1.$$ 

Hence, ALG 1 produces the updated working basis inverse.

5.2 Key Column Leaves The Basis

In this section, we present an algorithm for updating the working basis inverse to accomplish a switch between a key column and a nonkey column. That is, $\tilde{Q} = RQ^{-1}$ where $R$ is given by (13) and

$$Q^{-1} = U^1 \ldots U^m F^s \ldots F^1.$$  (20)

We wish to obtain $\tilde{Q}^{-1}$ in the same form as (20).

To accomplish this update, we begin with $\tilde{Q}^{-1} = R^1 \ldots U^m F^s \ldots F^1$. We apply Proposition 2 to $R^1$ creating the factorization $\tilde{Q}^{-1} = \tilde{U}^1 \tilde{R}^2 \tilde{U}^2 \ldots U^m F^s \ldots F^1$. We continue with the application of Proposition 2 until we obtain $\tilde{Q}^{-1} = \tilde{U}^1 \ldots \tilde{U}^k-1 R^k \tilde{U}^k \ldots U^m F^s \ldots F^1$. Proposition 2 does not apply to $R^k \tilde{U}^k$. However, a simple update would be to let $\tilde{U}^k = \ldots = \tilde{U}^m = 1$ and use the below factorization:

$$\tilde{Q}^{-1} = \tilde{U}^1 \ldots \tilde{U}^m R^k U^k \ldots U^m F^s \ldots F^1.$$  

LEFT FILE RIGHT FILE

This update simply involves application of Proposition 2 until it does not apply ($k = k^*$), and then shifting the remainder of the left file to the right file. We call this update the TYPE 1 UPDATE.

We will now give an update in which $R^k \tilde{U}^k \ldots U^m$ is modified as opposed to moving them to the right file. Let

$$E^k = R^k U^k$$

$\leftarrow$ row $k$

Then we define matrices $\tilde{U}^{k+1}$ and $E^{k+1}$ such that $E^k E^{k+1} = \tilde{U}^{k+1} E^{k+1}$. Following this procedure, $R^k U^k \ldots U^m$ can be replaced by $\tilde{U}^{k+1} \ldots U^m E^{k+1}$ so that

$$\tilde{Q}^{-1} = \tilde{U}^1 \ldots \tilde{U}^{k-1} \tilde{U}^k \ldots \tilde{U}^m E^{k+1} \ldots F^1.$$ 

Further, we define a row eta $\tilde{R}$ and a column eta $\tilde{F}$ such that $E^{n+1} = \tilde{R} \tilde{F}$. Therefore,

$$\tilde{Q}^{-1} = \tilde{U}^1 \ldots \tilde{U}^{k-1} \tilde{U}^k \ldots \tilde{U}^m \tilde{R} \tilde{F}^s \ldots F^1.$$  

LEFT FILE RIGHT FILE
We call this update the TYPE 2 UPDATE.
We now present a set of propositions which justify the TYPE 2 UPDATE.

**Proposition 7.**

Let

\[
\begin{align*}
\mathcal{U}^{i+1} &= \begin{bmatrix}
\eta_1 \\
\vdots \\
\eta_{i-1} \\
\eta_i \\
\eta_{i+1} \\
\eta_n \\
0
\end{bmatrix} \\
\mathcal{E}^P &= \begin{bmatrix}
\mu_1 \\
\vdots \\
\mu_{i-1} \\
\mu_i \\
\mu_{i+1} \\
\mu_n \\
0
\end{bmatrix}
\end{align*}
\]

and E\(^{P+1} =

\[
\begin{bmatrix}
\gamma_1 & \ldots & \gamma_i & \ldots & \gamma_{i+1} & \ldots & \gamma_n \\
\gamma_{i+1} & \ldots & \gamma_n & \gamma_1 & \ldots & \gamma_i & \ldots & \gamma_{i+1} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\gamma_n & \ldots & \gamma_1 & \ldots & \gamma_i & \ldots & \gamma_{i+1} & \ldots & \gamma_n \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \mu_i \\
\mu_i & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \mu_i \\
\mu_{i+1} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \mu_{i+1} \\
\mu_{i+1} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \mu_{i+1} \\
\mu_n & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \mu_n
\end{bmatrix}
\]

where \(\ell \neq \ell^*\) and \(\mu_i = 0\).

If

\[
\tilde{\mathcal{U}}^{p+1} = \begin{bmatrix}
\alpha_1 \\
\vdots \\
\alpha_{i-1} \\
\alpha_i \\
\alpha_{i+1} \\
\vdots \\
\alpha_n \\
0
\end{bmatrix} \\
\tilde{\mathcal{E}}^{p+1} = \begin{bmatrix}
\lambda_1 \\
\vdots \\
\lambda_{i-1} \\
\lambda_i \\
\lambda_{i+1} \\
\vdots \\
\lambda_n \\
0
\end{bmatrix}
\]

and \(E^{p+1} =

\[
\begin{bmatrix}
\mu_1 \\
\vdots \\
\mu_{i-1} \\
\mu_i \\
\mu_{i+1} \\
\vdots \\
\mu_n \\
0
\end{bmatrix}
\]

then \(E^{P+1} = \tilde{\mathcal{U}}^{p+1}E^{p+1}\).

The following proposition is used to replace the cross matrix \(E^{p+1}\) with a row \(\tilde{R}\) and a column \(\tilde{F}\).
Proposition 8.
Let
\[
E = \begin{bmatrix}
1 & 0 \\
\gamma_1 & \gamma_2 & \ldots & \gamma_n \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{\ell-1} & \gamma_{\ell} & \ldots & \gamma_n \\
0 & \mu_{\ell+1} & \ldots & 1
\end{bmatrix}
\]
\[
\bar{R} = \begin{bmatrix}
1 & 0 \\
\gamma_1 & \ldots & \gamma_{\ell-1} & X & \gamma_{\ell+1} & \ldots & \gamma_n \\
0 & 1
\end{bmatrix}
\]
and
\[
\bar{P} = \begin{bmatrix}
1 & \mu_1 & \ldots & \mu_{\ell-1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \mu_1 & \ldots & \mu_{\ell-1} & 1
\end{bmatrix}
\]
where \(X\) and \(Y\) are such that
\[
XY = \gamma_\ell - \sum_{i=1}^{\ell} \gamma_i \mu_i
\]
then \(E = \bar{R} \bar{F}\).
We now present the update algorithm for the case in which the \(\ell\)th column of \(T\) is being switched with the \(k\)th column of \(C\). Let \(\gamma = \alpha \bar{T}^{-1} C\).

**ALG 2: LU UPDATE FOR A KEY LEAVING COLUMN**

1. Set \(R_i^k \leftarrow I\).
   Set \(R_i^k[k] \leftarrow \gamma\).
   Set \(i \leftarrow 1\).
2. If \(i = k\), go to 4.
   Set \(\alpha \leftarrow R_i^k[k] U(i)\).
   Set \(R_i^k \leftarrow R_i^k - \alpha U(i)\).
   Set \(U(i) \leftarrow \alpha\).
   Set \(U_k \leftarrow 0\).
3. Set \(i \leftarrow i + 1\) and go to 2.
4. Set \(U^k \leftarrow I\).
5. Apply Proposition 7 to $\bar{E}^{i+1}$ to form $\bar{U}^{i+1}\bar{E}^{i+1}$. 
   Set $i \leftarrow i + 1$.
6. If $i < m$, go to 5.
7. Apply Proposition 8 to $E^m$ to obtain $\bar{P}^{1}$ where $X = 1$.
   At the completion of step 7 we have $\bar{G}^{-1} = \bar{U}^{1} \cdots \bar{U}^{m-1} \bar{P}^{1} \cdots \bar{F}^{1}$.

VI. COMPUTATIONAL EXPERIMENTATION

Three test problems were selected for the experiment. Sc205 is a stand-alone linear program which was generated by Ho and Loupe [12] and transformed into a network with side constraints. Gifford-Pinchot is a model of the Gifford-Pinchot National Forest [10] which has also been transformed into a network with side constraints. RAN is a randomly generated problem.

These problems were first solved and the pivot agenda was saved. That is, entering and leaving columns for each pivot were saved on a file. This file was then used by each code so that all three basis updates follow the same path to the optimum. The number of nonzeros required to represent $Q^{-1}$ at various points in the solution process is illustrated in Figures 1 and 2. For both problems, the LU Type 2 update dominated both the LU Type 1 update and the product-form code in terms of nonzeros in the inverse. The average core storage required for $Q^{-1}$ using the product-form update is approximately double that required for the best LU update.

Given the above results, we developed three specialized network with side constraints codes and computationally compared them with three general in-core LP systems and a special system for multicommodity network flow problems. All codes are written in FORTRAN and have not been tailored to either our equipment or our FORTRAN compiler. None of the codes were tuned for our problem set. A brief description of each code follows.

NETSIDE1, NETSIDE2 AND NETSIDE3 are our specialized network with side constraints systems. The first maintains $Q^{-1}$ in product form, while the second and third maintain $Q^{-1}$ in LU form using a Type 1 and Type 2 update, respectively. All use the Hellerman and Ranick [11] reversion routine. The working basis is reinveted every 60 iterations. The pricing routine uses a candidate list of size 6 with block size of 200.

MINOS [15] stands for "a Modular In-Core Nonlinear Optimization System" and is designed to solve problems of the following form:

minimize $f(x) + cx$
subject to: $Ax = b$
        $\ell \leq x \leq u$

where $f(x)$ is continuously differentiable in the feasible region. For this
study \( f(x) = 0 \) at all \( x \) and therefore none of the nonlinear subroutines were used for problem solution.

For linear programs, MINOS uses the revised simplex algorithm with all data and instructions residing in core storage. The basis inverse is maintained as an LU factorization using a Bartels-Golub update. The re inversion routine uses the Hellerman-Rarick [11] pivot agenda algorithm.

XMP is a library of FORTRAN subroutines which can be used to solve linear programs. The basis inverse is maintained in LU factored form. The pricing routine uses a candidate list of size 6 with two hundred columns being scanned each time the list is refreshed. The basis is reinverted every 50 iterations.

LISS stands for "Linear In-Core Simplex System" and is an in-core LP solver with the basis inverse maintained in product form. The re inversion routine is a modification of the work of Hellerman and Rarick [11]. The basis inverse is refactored every 50 iterations. A partial pricing scheme is used with 20 blocks.
MCNF stands for “Multicommodity Network Flow”. MCNF uses the primal partitioning algorithm also. The basis inverse is maintained as a set of rooted spanning trees (one for each commodity) and a working basis inverse in product form. This working basis inverse has dimension equal to the number of binding GUB constraints. A partial pricing scheme is used. Our computational experience is given in Table 1.

The row entitled GUB Constraints, gives the number of LP rows which correspond to “GUB Constraints”. The row, entitled “Binding GUB Constraints”, gives the number of GUB constraints met as equalities at optimality using MCNF. All runs were made on the CDC 6600 at Southern Methodist University using the FTN compiler with the optimization feature enabled.
Table 1 Comparison of Codes for Solving Multicommodity Network Flow Problems

<table>
<thead>
<tr>
<th>PROB DESC</th>
<th>Number</th>
<th>LP Rows</th>
<th>LP Con</th>
<th>% Network Rows</th>
<th>QUB Con</th>
<th>Binding QUB Const.</th>
<th>Number Nonzeros</th>
<th>Number Commodities</th>
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<td>100</td>
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<td>100</td>
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<tr>
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</tr>
<tr>
<td>% Network Rows</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
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<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>Number Commodities</td>
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<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

MINOS [31]

- **Scaled Time**: 94.4
- **Time (sec.)**: 340.3
- **Commodities**: 9
- **Pivots**: 25,022

XMP [28]

- **Scaled Time**: 63.2
- **Time (sec.)**: 146.3
- **Commodities**: 2
- **Pivots**: 2,592

LSS [2]

- **Scaled Time**: 364.8
- **Time (sec.)**: 458.8
- **Commodities**: 3
- **Pivots**: 2,991

MLOG [26]

- **Scaled Time**: 10.2
- **Time (sec.)**: 2.8
- **Commodities**: 1
- **Pivots**: 4,321

NETSO1

- **Scaled Time**: 230.4
- **Time (sec.)**: 230.4
- **Commodities**: 1
- **Pivots**: 886

NETSO12

- **Scaled Time**: 223.9
- **Time (sec.)**: 223.9
- **Commodities**: 1
- **Pivots**: 886

NETSO13

- **Scaled Time**: 232.4
- **Time (sec.)**: 232.4
- **Commodities**: 1
- **Pivots**: 886
Based on these results, we conclude that for lightly constrained multicommodity network flow problems:

(i) XMP and MINOS run at approximately the same speed.
(ii) NETSIDE1, NETSIDE2 and NETSIDE3 run at approximately the same speed, and
(iii) the three NETSIDE codes are approximately twice as fast as XMP and MINOS.

References


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