Network Modeling

Presentation Outline
- Value of network models
- Network model categories
- Pure network models
- Generalized networks
- Networks with side constraints

Why Study Network Flow Models?
- The most widely used optimization models today
- Popularity a result of:
  - Natural problem representation
  - Pictorial form
  - Integrality of solutions, in some cases
  - Fast, large-scale solution software

Natural Problem Representation
- Most LP problems have a substantial network component
- Application areas include:
  - Aircraft routing
  - Personnel planning
  - Plant location
  - Project management
  - Missile targeting
  - Production planning
  - Classroom scheduling
  - Cash flow management
  - Distribution-inventory
  - Evacuation planning

Pictorial Form
- Graphical format of network enhances
  - Modeling process
  - Communication
  - Acceptance by users
- Easy to understand and relate to
Solution Integrality

- “Pure” networks are LPs with
  - Integer-valued solutions
  - When problem data is integral
- AVOIDS challenging integer programming formulations
- Can model:
  - People hired, planes scheduled, ships built
  - Deliveries made, boxcars used, etc.

Efficient Solution Software

- Specialized procedures 100-150X faster than general LP
- Minimal data storage requirements
- Large-scale problems solved quickly
  - 10,000 constraints & 80,000 variables < 1 sec
  - 250,000 constraints, 2M variables in 30 PC sec
- Good for interactive applications

Network Solution Technology

- Recent merge example
  - Matching records from two large data files
  - Per US Treasury model
- Optimization times
  - 3.8 million variables in 9 seconds
  - 15 million in 10 min.
  - 44 million in 30 min.
- The result of extensive research by
  - F. Glover, R. Barr, and D. Klingman
- Pioneering work in network algorithms and NetForm modeling

Example Application Areas

- Large-scale personnel assignment
- Manpower planning
- Logistics and distribution
- Production planning and scheduling
- Mission planning & target selection
- Telecommunications network design

Categories of Network Flow Models

<table>
<thead>
<tr>
<th>Pure Networks</th>
<th>Generalized Networks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classic form</td>
<td></td>
</tr>
<tr>
<td>+ Linear side constraints</td>
<td></td>
</tr>
<tr>
<td>+ Discrete side constraints</td>
<td></td>
</tr>
</tbody>
</table>
**Network Modeling**

**Network Flow Models**

**Pure Networks**
- Shortest Path
- Maximum flow
- Assignment
- Transportation
- Transshipment

**Generalized Networks**
- Flows with gains and losses
- Generalized transportation
- Generalized transshipment

**Network Flow Models**

- Multi-objective
  - Goal programming targets
  - Preemptive priorities
- Linearly constrained
  - Multicommodity
  - Processing networks
  - Embedded networks
- Discretely constrained
  - Interval flow
  - Integer generalized
  - Generalized assignment

**Formulating Pure Networks**

**Pure Networks**
- Bipartite problems
  - Transportation
  - Assignment
  - Semi-assignment
- Transshipment
  - Path problems
  - Capacitated transshipment
  - Dynamic

**The Transportation Problem**

- A single commodity is distributed throughout this pure network
- Nodes
  - Location points
  - Supply of flow at source nodes (+)
  - Demand at sinks (-)

**Transportation Problems**

- Arrows
  - Connect pairs of nodes
  - Flow moves through arcs in direction of arrowhead
  - From the from-node to the to-node
  - Flow incurs a unit cost
Network Modeling

Objective Function

• Objective: pattern of routing commodity that minimizes total cost
• Solutions observe flow conservation at nodes
• Example solution:

```
   +50  DFW
   +150  ATL
   +100  NY
   -100  LA
   +100  DEN

   50 = flow from DFW to NY
   100 = flow from NY to LA
   100 = flow from NY to DEN
```

Linear Programming Model

• Arcs are variables:
  \[ x_{ij} = \text{units of flow from node } i \text{ to node } j \text{ on arc } (i,j) \]
• Nodes are constraints:
  \[ \text{Flow supplied at node } + \text{flow arriving on arcs} = \text{flow demanded} + \text{flow leaving on arcs} \]

\[ i \rightarrow C_{ij} \times x_{ij} \rightarrow j \]

LP Formulation

```
  SUPPLY NODES
  TRANSPORTATION ROUTES
  DEMAND NODES

  +180
  +120
  +100
  +160
  +150
  +100
  -100
  -150
  -100
  -100
```

Electric Co-op Rate Model

Minimize total MW-KM

```
  Disgorgement
  from Power Sources
  "\[ \text{Distance Traveled} \]

  +10
  +20
  +15
  -10
  -17
  -13
```

Assignment Problems

• Transportation problem with unit requirements
• No partial assignments made
• Dummy nodes for imbalances

```
  Worker  Job
  1  A  1
  1  B  1
  1  C  1
  1  D  1
  2  E  1
  2  F  1
  2  G  1

  Cost to hire to perform job
```

Semi-Assignment Problems

• Combination of assignment and transportation
• Example: medical student residency assignment
• Dummy nodes for imbalances

```
  Student  Hospitals & number of positions available
  1  A  1
  2  B  2
  1  C  1
  2  D  2
  1  E  1
  2  F  2

  Cost = student ranking + hospital ranking
```
Transshipment Networks

- Transshipment nodes
  - Arcs enter and leave
- Transshipment sources and sinks

Costs not shown

Longest Chain: PERT/CPM

- Nodes = activities
  - Arcs show activity precedence order
- Cost = completion time of from-node activity
  - Find: critical activity path and slack

Shortest Path Problem

- Costs = distances or travel time
  - 1-to-1 or 1-to-all

Job-Scheduling Problem

- Four jobs are to be processed: J₁ – J₄
  - Each job Jᵢ
    - Requires time Tᵢ to complete
    - Has cost Cᵢ(𝑡) to complete at time 𝑡
    - Uninterrupted
  - Find min cost job ordering

Job Scheduling

Adding Bounds

- Capacitated networks permit upper and lower bounds on each arc’s flow
  - Correspond to: \( L_{ij} \leq x_{ij} \leq U_{ij} \)

Nonlinear costs and completion-time functions possible
Network Modeling

Max Flow Problem

![Max Flow Problem Diagram]

Flow units = millions of barrels of oil per hour

Production-Distribution

![Production-Distribution Diagram]

Logistics Network

![Logistics Network Diagram]

Dynamic Networks

- Involve multiple time periods
- Typical structure:
  - Series of one-period network models
  - Connected by “inventory” and other inter-temporal arcs

Multi-Period Production

- Regular and overtime production capacity per period
- Monthly demand
- Costs:
  - Regular production
  - Overtime production
  - Inventory

Production-Distribution

![Multi-Period Production Diagram]
**Network Modeling**

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**Piecewise Linear Costs**

- Parallel arcs can model piece-wise linear:
  - Convex costs
  - Concave revenue
- Example:
  - where \( C_1 < C_2 < C_3 \)

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**Goal Programming**

- Set target flow value on one arc
- Add parallel arcs with penalties for missing target
- Minimize total penalties

---

**Manpower Planning Model**

- When several objectives are present
- Preemptive priorities consider objectives in priority order
  - \( P_1 \) = highest priority
  - \( P_2 \) = second-highest, etc.
- Exploit alternate optima of \( P_1 \) objective to optimize \( P_2 \) as much as possible

---

**Multiple Objectives, Priorities**

- Set target flow value on one arc
- Add parallel arcs with penalties for missing target
- Minimize total penalties
**Personnel Assignment**

- Individual assignments determined by:
  - Eligibility for assignment
  - Services quotas for assignment type and group
  - Multiple objectives and priorities
- NetForm permits solution of massive problems

**Generalized Networks**

- "Flows with gains and losses"
- For every unit of flow that leave node $i$ on arc $(i,j)$, $M_{ij}$ units arrive at node $j$
- If $M_{ij}$:
  - $< 1$, flow is attenuated (lossy)
  - $> 1$, flow is amplified (gain)
  - $= 1$, same as a pure-network arc
- Costs are applied to the incoming arc flow, $x_{ij}$ prior to multiplier application
- Applications
  - Change flow levels
  - Change flow units

**Investment Model**

Maximize flow:

For two periods:

1. 1-period bank loan @ 10% interest
2. 2-period investment with 11% return
3. 1-period investment with 8% return

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<tr>
<td>0</td>
<td>$100 \times M$</td>
<td>$M = 1.08$</td>
<td>$M = 1.09$</td>
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<tr>
<td>2</td>
<td>$200 \times M$</td>
<td></td>
<td>$M = 1.11$</td>
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Electrical line losses:

$X = 100, M = 0.98$
$X = 100, M = 0.95$
$X = 100, M = 0.95$

Change in level of flow:

- Funds available at start of period 1
- Initial investment with 8% return
- Investment after two periods

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Multipliers Change Flow Units

- Conversion rates

\[
\begin{align*}
M &= 1.47 \\
M &= 9.21 \\
M &= 0.0094
\end{align*}
\]

Arbitrage Model

- Flows = funds converted between currencies
- Multipliers = current exchange rates, including any conversion fees
- When does arbitrage exist?

\[
\begin{align*}
M &= 1.47 \\
M &= 0.90 \\
M &= 0.90 \\
M &= 1.47 \\
M &= 1.58 \\
M &= 1.58 \\
M &= 4.70 \\
M &= 4.70
\end{align*}
\]

Retailing Model

Networks with Linear Side Constraints

- A network formulation with 1+ non-network constraints
- Integral flows not guaranteed
- Transform to pure network, if possible

Equivalent Networks

- Subset of arcs out of a node:
Network Modeling

Processing Networks

• Proportional flows required:

\[
\begin{bmatrix}
0.4 \\
0.1 \\
0.5
\end{bmatrix}
\]

Multicommodity Networks

• Copy of network for each of \( k \) commodities
• Upper bound on sum of flows on corresponding arcs

Telecom Network Design

• Commodity = an origin-destination demand
• Network span capacities & costs

Networks with Discrete Side Constraints

Fixed-Charge Networks

• Arcs may have costs that are:
  – Variable with flow
  – Fixed, if flow exists
• Fixed cost, \( F \), is independent of amount of flow

Facility Location Model
Applications

- Setup, startup, changeover, expansion costs
- Binary decisions:
  - Purchase or lease of equipment
  - Personnel hiring
  - Build vs. no-build
- Location and network design problems
- Concave cost functionals

Concave Cost Functionals

- Piece-wise linear separable concave functionals can be expressed as a fixed-charge problem
- Consider an arc with the following cost structure:

Concave Costs

- This can be expressed by a series of variables
- One per segment

\[ F_1, C_1 \]
\[ F_2, C_2 \]
\[ F_3, C_3 \]