Exercise 1.1. Formulation exercise.

Summary. A set of seven investments are under consideration, each of which are available on an all-or-nothing basis. Formulate different types of restrictions on the portfolio of investments to be made.

Variables. The variables used, and their interpretations, are:

\[ x_i = 0/1 \equiv \text{don’t/do select investment } i, i = 1, \ldots, 7 \]
\[ y = 0/1, \text{ for choosing between two investment groups} \]

Formulations. Each formulation is considered independently, without regard for interactions with other constraints.

(i) \[ \sum_{i=1}^{7} x_i \leq 6 \]
(ii) \[ \sum_{i=1}^{7} x_i \geq 1 \]
(iii) \[ x_1 \leq (1 - x_3) \text{ or } x_1 + x_3 \leq 1 \]
(iv) \[ x_4 \leq x_2 \]
(v) \[ x_1 = x_5 \]
(vi) The two constraints: \[ x_1 + x_2 + x_3 \geq y \text{ and } x_2 + x_4 + x_5 + x_6 \geq (1 - y) \]

Exercise 1.7. Formulation exercise.

Summary. A nine-week production-planning model for scheduling a set of jobs of varying lengths and weekly labor requirements, subject to a weekly labor restriction, plus additional interactions between the jobs.

Assumptions. Once begun, a job cannot be interrupted. A job can be started in any week for which it can be completed before the end of the planning period. Workers are interchangeable across all jobs. The following data is available:

\[ p_i : \text{the number of weeks required to perform job } i \]
\[ l_{i,u} : \text{number of workers required in the } u\text{-th week of executing job } i \]
\[ L_t : \text{workers available in week } t \]
**Variables.** The variables and sets used, and their interpretations, are:

- $I$: The set of all jobs to be scheduled
- $T$: The set of weeks to be planned (e.g. $\{1, \ldots, 9\}$)
- $B_i$: the set of possible starting weeks for job $i \in I$,
  $$B_i = \{t \in T : (t + p_i - 1) \in T\}$$
- $x_{it}$: 0/1 $\equiv$ don’t/do start job $i \in I$ in week $t \in B_i$
- $w_t$: number of workers used in week $t \in T$
- $z$: maximum number of workers in any single week
- $a_{it}$: 0/1 $\equiv$ job $i$ is not/is active in week $t$

**Formulations.** Each formulation is considered independently, without regard for interactions with other parts’ constraints.

1. Feasible solution finder. No optimization required, just a feasible schedule, hence an objective whose value is a constant. (Any bounded term would suffice.)

   $\begin{align*}
   \text{maximize} & \quad \sum_{i \in I} \sum_{t \in B_i} x_{it} \\
   \text{subject to:} & \quad \sum_{t \in B_i} x_{it} = 1, \forall i \in I \\
   & \quad \sum_{i \in I} \sum_{u \in B_i: u \leq t \leq (u + p_i)} l_{i,(t-u+1)} x_{iu} = w_t, \forall t \in T \\
   & \quad w_t \leq L_t, \forall t \in T \\
   & \quad x_{it} \in \{0,1\}, \forall i \in I, t \in B_i
   \end{align*}$

   For this base model, constraints (2) ensure that each job is scheduled to begin in one of the possible weeks, (3) calculate the weekly labor utilization, and (4) enforces the weekly labor capacity restrictions. The summation in (3) includes all jobs that might be active during a given week, based on starting weeks and execution time required.

2. To minimize the greatest single weekly workload, constraints (7) are added to the base model in part 1, and the objective (1) is replaced by (6), as follows:

   $\begin{align*}
   \text{Minimize} & \quad z \\
   \text{subject to:} & \quad w_t \leq z, \forall t \in T
   \end{align*}$

   While minimizing, the variable $z$ will be limited by the largest $w_t$ value.
3. Job 1 must start at least two weeks before job 3. To formulate, certain week combinations must be rendered infeasible. The base model or that of part 2 should be augmented by the following constraints:

$$\sum_{u=1}^{t-2} x_{1u} \geq x_{3t}, t = 3, \ldots, 9$$

$$x_{31} = x_{32} = 0$$

4. Jobs 4 must start not later than one week after job 5.

$$\sum_{u=1}^{t+1} x_{4u} \geq x_{5t}, t = 1, \ldots, 8$$

5. Jobs 1 and 2 may not be simultaneously active (e.g., in progress).

$$a_{1t} \geq \max\{1,(t-p_1+1)\} \sum_{u=t}^{\max\{1,(t-p_1+1)\}} x_{1u}, \forall t \in T$$

$$a_{2t} \geq \max\{1,(t-p_2+1)\} \sum_{u=t}^{\max\{1,(t-p_2+1)\}} x_{2u}, \forall t \in T$$

$$a_{1t} + a_{2t} \leq 1, \forall t \in T$$