Integer Programming Exercises Solutions

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1. For the model stated:

   (a) Maximize:

   $$x_1 + 1/2x_2$$

   $$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 12 \\ 5x_1 + x_4 &= 10 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

   (b)

   $$\mathbf{x}_{B1} = (x_1, x_2), \mathbf{B}_1 = \begin{pmatrix} 3 & 2 \\ 5 & 0 \end{pmatrix}$$

   $$\mathbf{x}_{B2} = (x_1, x_4), \mathbf{B}_2 = \begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix}$$

   (c) $\mathbf{x}_{B1}$: Solving

   $$\begin{aligned} 3x_1 + 2x_2 &= 12 \\ 5x_1 &= 10 \end{aligned}$$

   yields $x_1 = 2, x_2 = 3, \mathbf{x}_{B1} = (2, 3)$.
   $\mathbf{x}_{B2}$: Solving

   $$\begin{aligned} 3x_1 &= 12 \\ 5x_1 + x_4 &= 10 \end{aligned}$$

   yields $x_1 = 4, x_4 = -10, \mathbf{x}_{B2} = (4, -10)$.

   (d) $\mathbf{x}_{B1}$ is feasible, $\mathbf{x}_{B2}$ is infeasible, since $x_4 < 0$.

   (e)

   $$\mathbf{x}_{B1}: \begin{pmatrix} 3 \\ 5 \end{pmatrix} x_1 + \begin{pmatrix} 2 \\ 0 \end{pmatrix} x_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} 2 + \begin{pmatrix} 2 \\ 0 \end{pmatrix} 3 = \begin{pmatrix} 6+6 \\ 10+0 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \end{pmatrix}$$

   $$\mathbf{x}_{B2}: \begin{pmatrix} 3 \\ 5 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_4 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} 4 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (-10) = \begin{pmatrix} 12+0 \\ 20-10 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \end{pmatrix}$$

2. Continuing from the previous problem.

(a) The tableaus are:

Tableau 1			1	0.5	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4
0	x_3	12	3	2	1	0
0	x_4	10	(5)	0	0	1
c_j -	z_j	0	1	1/2	0	0

Entering variable is x_1, ratio test finds that x_4 leaves at $\theta = 2$.

Tableau 2			1	0.5	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4
0	x_3	6	0	(2)	1	-0.6
1	x_1	2	1	0	0	0.2
c_j -	z_j	-2	0	0.5	0	-0.2

Entering variable is x_2, x_3 leaves at $\theta = 3$, creating an optimal solution.

Tableau 3			1	0.5	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4
0.5	x_2	3	0	1	0.5	-0.3
1	x_1	2	1	0	0	0.2
c_j -	z_j	-3.5	0	0	-0.25	-0.05

(b) $\mathbf{B}^{-1} = \begin{pmatrix} 1/2 & -3/10 \\ 0 & 1/5 \end{pmatrix}$

(c) $\mathbf{c}_B \mathbf{B}^{-1} = (1/2, 1) \begin{pmatrix} 1/2 & -3/10 \\ 0 & 1/5 \end{pmatrix} = (1/4, 1/20)$

$$z_1 = (1/4, 1/20) \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3/4 + 1/4 = 1, c_1 - z_1 = 0$$

$$z_2 = (1/4, 1/20) \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 1/2, c_2 - z_2 = 0$$

$$z_3 = (1/4, 1/20) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1/4, c_3 - z_3 = -1/4$$

$$z_4 = (1/4, 1/20) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1/20, c_4 - z_4 = -1/20$$

(d) $A_4 = -3/10 A_2 + 1/5 A_1 = -3/10 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1/5 \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -3/5 + 3/5 \\ 0 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

3.

Tableau 1			2	3	0	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5
0	x_3	1	-5	(1)	1	0	0
0	x_4	5	1	2	0	1	0
0	x_5	4	2	1	0	0	1
c_j - z_j		0	2	3	0	0	0

x_2 replaces x_3 at $\theta = 1$.

Tableau 2			2	3	0	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5
3	x_2	1	-5	1	1	0	0
0	x_4	3	(11)	0	-2	1	0
0	x_5	3	7	0	-1	0	1
c_j - z_j		-3	17	0	-3	0	0

x_1 replaces x_4 at $\theta = 3/11$.

Tableau 3			2	3	0	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5
3	x_2	26/11	0	1	1/11	5/11	0
2	x_1	3/11	1	0	-2/11	1/11	0
0	x_5	12/11	0	0	3/11	-7/11	1
c_j - z_j		-84/11	0	0	1/11	-17/11	0

x_3 replaces x_5 at $\theta = 4$.

Tableau 4			2	3	0	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5
3	x_2	2	0	1	0	2/3	-1/3
2	x_1	1	1	0	0	-1/3	2/3
0	x_3	4	0	0	1	-7/3	11/3
c_j - z_j		-8	0	0	0	-4/3	-1/3

Optimal.

4. (a) Completed tableau:

Tableau			1	1	0	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5
1	x_1	1	1	0	0	1/5	1/5
1	x_2	2	0	1	0	3/5	-2/5
0	x_3	1	0	0	1	-7/5	3/5
c_j - z_j		-3	0	0	0	-4/5	1/5

(b) Current solution is not optimal, since $c_5 - z_5 > 0$.

(c) The representation of A_4 is $\tilde{A}_4 = \begin{pmatrix} 1/5 \\ 3/5 \\ -7/5 \end{pmatrix}$, from the tableau.

5. (Note: variable x_4 shown below is unnecessary if x_3 is used in forming the identity matrix in the initial solution.

Tableau 1			2	5	3	0	-M
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5
0	x_4	5	1	2	1	1	0
-M	x_5	4	(2)	-1	0	0	1
c_j -	z_j	4M	2M+2	-M+5	3	0	0

Tableau 2			2	5	3	0	-M
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5
0	x_4	3	0	(2.5)	1	1	-.5
2	x_1	2	1	-.5	0	0	.5
c_j -	z_j	-4	0	6	3	0	-M-1

Tableau 3			2	5	3	0	-M
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5
5	x_2	1.2	0	1	(.4)	.4	-.2
2	x_1	2.6	1	0	.2	.2	.4
c_j -	z_j	-11.2	0	0	.6	-2.4	-M+.2

Tableau 4			2	5	3	0	-M
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5
3	x_3	3	0	2.5	1	1	-.5
2	x_1	2	1	-.5	0	0	.5
c_j -	z_j	-13	0	-1.5	0	-3	-M+.5

Optimal with $\mathbf{x} = (2, 0, 3, 0)$ and objective function value = -13.

6.

Tableau I-1			0	0	0	-1	-1
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5
-1	x_4	6	3	2	1	1	0
-1	x_5	4	2	1	(5)	0	1
c_j -	z_j	10	5	3	6	0	0

Tableau I-2			0	0	0	-1	-1
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5
-1	x_4	5.2	2.6	1.8	0	1	-1.2
0	x_3	.8	(.4)	.2	1	0	.2
c_j -	z_j	5.2	2.6	1.8	0	0	-1.2

Arbitrarily breaking the ratio-test tie by designating x_3 as the leaving variable.

Tableau I-3			0	0	0	-1	-1
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5
-1	x_4	0	0	(.5)	-6.5	1	-1.5
0	x_1	2	1	.5	2.5	0	.5
c_j -	z_j	0	0	.5	-6.5	0	-2.5

Tableau I-4			0	0	0	-1	-1
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5
0	x_2	0	0	1	-13	2	-3
0	x_1	2	1	0	9	-1	2
c_j -	z_j	0	0	0	0	-1	-1

Optimal for Phase I.

Tableau II-1			1	.25	5	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5
.25	x_2	0	0	1	-13	2	-3
1	x_1	2	1	0	9	-1	2
c_j -	z_j	-2	0	0	-.75	.5	-1.25

Optimal for Phase II. Note: x_4 and x_5 are artificials, and are not eligible for pivoting.

7. (a) We know that $\mathbf{B}^{-1} = (\tilde{A}_4, \tilde{A}_5, \tilde{A}_6)$ because these are the slack columns forming the problem's identity matrix. Hence

$$\tilde{A}_1 = \mathbf{B}^{-1}A_1 = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\tilde{A}_2 = \mathbf{B}^{-1}A_2 = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\tilde{A}_3 = \mathbf{B}^{-1}A_3 = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\mathbf{x}_B = \begin{pmatrix} x_2 \\ x_1 \\ x_6 \end{pmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

$$\mathbf{c}_B = (c_2, c_1, c_6) = (3, 2, 0)$$

$$z_1 = \mathbf{c}_B\tilde{A}_1 = (3,2,0)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2, c_1 - z_1 = 0$$

$$z_2 = \mathbf{c}_B\tilde{A}_2 = (3,2,0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 3, c_2 - z_2 = 0$$

$$z_3 = \mathbf{c}_B\tilde{A}_3 = (3,2,0)\begin{pmatrix} 1 \\ -1 \\ 6 \end{pmatrix} = 1, c_3 - z_3 = 3$$

$$z_4 = \mathbf{c}_B\tilde{A}_4 = (3,2,0)\begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix} = 0, c_4 - z_4 = 0$$

$$z_5 = \mathbf{c}_B\tilde{A}_5 = (3,2,0)\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 1, c_5 - z_5 = -1$$

$$z_6 = \mathbf{c}_B\tilde{A}_6 = (3,2,0)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0, c_6 - z_6 = 0$$

$$z = \mathbf{c}_B\mathbf{x}_B = (3,2,0)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2$$

Therefore, the tableau is:

Tableau			2	3	4	0	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5	x_6
3	x_2	0	0	1	1	2	-1	0
2	x_1	1	1	0	-1	-3	2	0
0	x_6	3	0	0	2	-2	1	1
c_j -	z_j	-2	0	0	3	0	-1	0

(b) The solution is not optimal, since nonbasic x_3 is pivot eligible.

8. Let $x_1, x_2 =$ the number of shirt types A and B, respectively, to produce.

(a) Maximize

$$9x_1 + 8x_2$$

subject to:

$$\begin{aligned} x_1 + 2x_2 &\leq 20 \\ 3x_1 + 4x_2 &\leq 60 \\ 5x_1 + 3x_2 &\leq 90 \\ x_1, x_2 &\geq 0 \end{aligned}$$

In standard form, the constraints become:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 20 \\ 3x_1 + 4x_2 + x_4 &= 60 \\ 5x_1 + 3x_2 + x_5 &= 90 \\ x_1, \ldots, x_5 &\geq 0 \end{aligned}$$

The revised simplex tableaus, which maintain only $\mathbf{B}^{-1}, \mathbf{x}_B, \mathbf{u}, z$, are:

Tableau 1						0	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5	x_6
0	x_3	20				1	0	0
0	x_4	60				0	1	0
0	x_5	90				0	0	1
c_j -	z_j	0				0	0	0

Using the duals $\mathbf{u} = (0, 0, 0)$ and the original data from $\mathbf{A}$ to compute each $c_j - z_j = c_j - \mathbf{u}A_j, x_1$ has the highest pivot-eligible marginal value of 9. Using $\tilde{A}_1 = \mathbf{B}^{-1}A_1 = (1, 3, 5)$, the ratio test reveals that x_5 should leave the basis. Pivoting yields:

Tableau 2						0	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5	x_6
0	x_3	2				1	0	$-1/5$
0	x_4	6				0	1	$-3/5$
9	x_1	18				0	0	$1/5$
c_j -	z_j	-162				0	0	$-9/5$

Now that $\mathbf{u} = (0, 0, 1.8), c_2 - z_2 = 2.6$ is the largest positive marginal value. Bringing in x_2 and using $\tilde{A}_2 = \mathbf{B}^{-1}A_2 = (7/5, 11/5, 3/5)$, the ratio test shows that x_3 leaves the basis. Pivoting,

Tableau 3						0	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5	x_6
8	x_2	$10/7$				$5/7$	0	$-1/7$
0	x_4	$20/7$				$-11/7$	1	$-2/7$
9	x_1	$120/7$				$-3/7$	0	$2/7$
c_j -	z_j	$-1160/7$				$-13/7$	0	$-10/7$

With $\mathbf{u} = (13/7, 0, 10/7)$, no variables are pivot eligible and the optimal solution is $\mathbf{x}_B = (x_2, x_4, x_1) = (10/7, 20/7, 120/7)$, with a solution value of $1160/7 = 165.714$.

(b) Minimize

$$20u_1 + 60u_2 + 90u_3$$

subject to:

$$\begin{aligned} u_1 + 3u_2 + 5u_3 &\geq 9 \\ 2u_1 + 4u_2 + 3u_3 &\geq 8 \\ u_1, u_2, u_3 &\geq 0 \\ u_1, u_2, u_3 & \quad \text{unrestricted in sign (redundant)} \end{aligned}$$

(c) The optimal dual solution is: $\mathbf{u} = (13/7, 0, 10/7)$.

9. (a) Rewriting the problem in standard form:
Maximize

$$-4x_1 - 5x_2$$

subject to:

$$\begin{aligned} -x_1 - 4x_2 + x_3 &= -5 \\ -3x_1 - 2x_2 + x_4 &= -7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

This reveals a dual-feasible initial basic solution: $\mathbf{x}_B = (x_3, x_4) = (-5, -7)$ to which the Dual Method can be applied.

Tableau 1			-4	-5	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4
0	x_3	-5	-1	-4	1	0
0	x_4	-7	(-3)	-2	0	1
c_j -	z_j	0	-4	-5	0	0

Choosing x_4 to leave the basis, the ratio test finds that x_1 should enter the basis (min ratio of $\phi = \min\{4/3, 5/2\} = 4/3$). Pivoting on -3,

Tableau 2			-4	-5	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4
0	x_3	-8/3	0	(-10/3)	1	0
-4	x_1	7/3	1	2/3	0	-1/3
c_j -	z_j	28/3	0	-7/3	0	-4/3

Selecting primal infeasible x_3 is to leave the basis, then x_2 must enter with $\phi = \min\{7/10, 4/1\} = 7/10$. Pivoting,...

Tableau 3			-4	-5	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4
-5	x_2	8/10	0	1	-3/10	1/10
-4	x_1	18/10	1	0	2/10	-4/10
c_j -	z_j	112/10	0	0	-7/10	-11/10

The LP solution is both primal and dual feasible, hence optimal.

10. Note that the answers do not always verify because of the lack of sufficient significant dignits in the printed tableau.

    (a) (See Figure 1.) The values in the **Decision Variable Summary** are drawn directly from the tableau. The basic solution gives the optimal values of x_1 and x_2 and, since it is nonbasic, x_3 is 0. The $c_j - z_j$ row provides the marginal values for all variables.

    For the **Constraint Summary**, slack x_4 (`SLK:CONST1`) is basic at 125, the other slacks are nonbasic at 0, and the marginal values are the $c_j - z_j$ values for the slacks.

    Sensitivity analysis checks the ranges on the c_j for which all $c_j - z_j$ remain ≤ 0. For c_1, all nonbasics are tested to determine the range over which the current basis will remain optimal (all $c_j - z_j \leq 0$).

$$\begin{aligned} c_3 - z_3 = 20 - [.688(0) + 1.938(40) - 1.625c_1] &\leq 0 \\ c_1 &\leq 35.397 \end{aligned}$$

$$\begin{aligned} c_5 - z_5 = 0 - [.125(0) + .625(40) - .75c_1] &\leq 0 \\ c_1 &\leq 33.33 \end{aligned}$$

$$\begin{aligned} c_6 - z_6 = 0 - [-.019(0) - .044(40) + .062c_1] &\leq 0 \\ c_1 &\geq 28.38 \end{aligned}$$

*** DECISION VARIABLE SUMMARY ***

Variable	Opt Value	Unit PROFIT	Marg Value
1. X1	250	30.000	0
2. X2	625	40.000	0
3. X3	0	20.000	-8.75

*** CONSTRAINT SUMMARY ***

Constraint	Rel	RHS	Slack/Surplus	Marginal Value
1. CONST1	<=	1000.000	125	0
2. CONST2	<=	8000.000	0	-2.5
3. CONST3	<=	100000.000	0	-0.125

*** SENSITIVITY OF THE PROFIT VALUES ***

Variable	Current OF	Upper Limit	(Becomes+)	Lower Limit	(Becomes+)
1. X1	30.000	33.333	X5(SLK:CONST2)	28.38	X3
2. X2	40.000	42.272	X6(SLK:CONST3)	36	X5
3. X3	20.000	28.75	X3	-INF	--

*** RANGING THE RIGHT-HAND SIDE ***

Constraint	RHS	Upper Limit	(Becomes 0)	Lower Limit	(Becomes 0)
1. CONST1	1000.00	+INF		900	X4
2. CONST2	8000.00	8266.67	X1	7040	X2
3. CONST3	100000.00	105263	X4	96774	X1

Figure 1: Completed Printout

Hence if $c_1 < 28.38, x_6$ becomes pivot eligible or if $c_1 > 33.33, x_5$ becomes pivot eligible, yielding the range of sensitivity to be $28.38 \leq c_1 \leq 33.33$ for the current solution to remain optimal. Similarly for c_2:

$$\begin{aligned} c_3 - z_3 = 20 - [.688(0) + 1.938c_2 - 1.625(30)] &\leq 0 \\ c_2 &\geq -14.834 \end{aligned}$$

$$\begin{aligned} c_5 - z_5 = 0 - [.125(0) + .625c_2 - .75(30)] &\leq 0 \\ c_2 &\geq 36 \end{aligned}$$

$$\begin{aligned} c_6 - z_6 = 0 - [-.019(0) - .044c_2 + .062(30)] &\leq 0 \\ c_2 &\leq 42.272 \end{aligned}$$

giving the range of sensitivity to be $36 \leq c_2 \leq 42.272$. For nonbasic x_3, it has no lower limit to remain nonbasic, and $z_3 = c_3 - (c_j - z_j) = 28.75$ provides the upper limit on c_3, at which point x_3 becomes pivot-eligible and, at optimality, basic.

To **range the right-hand-side values**, we consider values of b_i for which the basic variables remain nonnegative, using the relationship:

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$$

Using the optimal basis and $\mathbf{B}^{-1}$ taken from the tableau, we evaluate b_1 as follows.

$$\mathbf{x}_B = \begin{pmatrix} x_4 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & .125 & -.019 \\ 0 & .625 & -.044 \\ 0 & -.75 & .062 \end{pmatrix} \begin{pmatrix} b_1 \\ 8000 \\ 100,000 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} b_1 + .125(8000) - .019(100,000) \\ 0 + .625(8000) - .044(100,000) \\ 0 - .75(8000) + .062(100,000) \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} b_1 - 900 \\ 600 \\ 800 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence, b_1 must remain ≥ 900 to ensure that $x_4 \geq 0$. Evaluating b_2 similarly yields:

$$\mathbf{x}_B = \begin{pmatrix} x_4 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & .125 & -.019 \\ 0 & .625 & -.044 \\ 0 & -.75 & .062 \end{pmatrix} \begin{pmatrix} 1000 \\ b_2 \\ 100,000 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1000 + .125b_2 - .019(100,000) \\ 0 + .625b_2 - .044(100,000) \\ 0 - .75b_2 + .062(100,000) \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} b_2 - 7200 \\ b_2 - 7040 \\ -b_2 + 8266.67 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore the range $7040 \leq b_2 \leq 8266.67$ is defined by basic variables x_2 and x_1, respectively. Finally considering b_3,

$$\mathbf{x}_B = \begin{pmatrix} x_4 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & .125 & -.019 \\ 0 & .625 & -.044 \\ 0 & -.75 & .062 \end{pmatrix} \begin{pmatrix} 1000 \\ 8000 \\ b_3 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1000 + .125(8000) - .019b_3 \\ 0 + .625(8000) - .044b_3 \\ 0 - .75(8000) + .062b_3 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} -b_3 + 105,263 \\ -b_3 + 113,636 \\ b_3 - 96,774 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Giving the range: $96,774 \leq b_3 \leq 105,263$.

(b) Using

$$\mathbf{x}_B = \begin{pmatrix} x_4 \\ x_2 \\ x_1 \end{pmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 1 & .125 & -.019 \\ 0 & .625 & -.044 \\ 0 & -.75 & .062 \end{pmatrix} \begin{pmatrix} 1200 \\ 7000 \\ 90,000 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1200 + .125(7000) - .019(90,000) \\ 0 + .625(7000) - .044(90,000) \\ 0 - .75(7000) + .062b_3 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 365 \\ 415 \\ 330 \end{pmatrix}$$

Since $\mathbf{x}_B \geq \mathbf{0}$, the solution is feasible.

(c) Since the current solution violates the new constraint, an expanded tableau must be generated. Restating the constraint as

$$-x_1 + x_2 + x_7 = 100$$

and adding x_7 to the basis, the new basis inverse, $\mathbf{B'}^{-1}$, is computed as:

$$\mathbf{B'}^{-1} = \left(\begin{array}{c|c} \mathbf{B}^{-1} & \mathbf{0} \\ \hline -(a_{m+1,B1}, a_{m+1,B2}, \ldots, a_{m+1,Bm})\mathbf{B}^{-1} & 1 \end{array}\right)$$

$$= \left(\begin{array}{c|c} \mathbf{B}^{-1} & \mathbf{0} \\ \hline -(0,1,-1)\mathbf{B}^{-1} & 1 \end{array}\right)$$

where

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & .125 & -.019 \\ 0 & .625 & -.044 \\ 0 & -.75 & .062 \end{pmatrix}$$

yielding

$$\mathbf{B'}^{-1} = \begin{pmatrix} 1 & .125 & -.019 & 0 \\ 0 & .625 & -.044 & 0 \\ 0 & -.75 & .062 & 0 \\ 0 & -1.375 & 0.106 & 1 \end{pmatrix}$$

The new inverse can be used to generate new values for $\mathbf{x}_B = \mathbf{B'}^{-1}\mathbf{b'}$ and $\tilde{A}_3 = \mathbf{B'}^{-1}A_3$, shown in the tableau:

Tableau 3			30	40	20	0	0	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	x_4	125	0	0	.688	1	.125	$-.019$	0
40	x_2	625	0	1	1.938	0	.625	$-.044$	0
30	x_1	250	1	0	-1.625	0	$-.75$	.062	0
0	x_7	-300	0	0	-3.58	0	(-1.375)	.106	1
c_j -	z_j	-32500	0	0	-8.75	0	-2.5	$-.125$	0

The solution is dual feasible but primal infeasible. Requiring x_7 to leave the basis results in x_5 being the entering variable ($\phi = 2.5/1.375 = 1.8$). Pivoting yields the optimal solution.[1]

Tableau 4			30	40	20	0	0	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	x_4	92.72	0	0	.364	1	0	$-.009$	.091
40	x_2	488.63	0	1	.318	0	0	.005	.455
30	x_1	413.64	1	0	.318	0	0	.005	.545
0	x_5	218.18	0	0	2.591	0	1	$-.077$	.727
c_j -	z_j	-31954	0	0	-2.27	0	0	$-.318$	-1.818

11. Rewriting the problem in standard form:

Maximize

$$-4x_1 - 5x_2$$

subject to:

$$\begin{aligned} -x_1 - 4x_2 + x_3 &= -5 \\ -3x_1 - 2x_2 + x_4 &= -7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

[1]The true $\mathbf{x}_B = (100, 500, 400, 200)$. The large errors above are due to truncation in the available data.

Solving with the Dual Method yields:

Tableau 3			-4	-5	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4
-5	x_2	8/10	0	1	-3/10	1/10
-4	x_1	18/10	1	0	2/10	-4/10
c_j -	z_j	112/10	0	0	-7/10	-11/10

The LP solution is both primal and dual feasible, hence optimal.

Solving the integer problem using Gomory's cuts. Using x_1 to generate the cut,[2] we require that $8/10 - 2/10x_3 - 6/10x_4 \leq 0$, as shown.

Tableau 3			-4	-5	0	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5
-5	x_2	8/10	0	1	-3/10	1/10	0
-4	x_1	18/10	1	0	2/10	-4/10	0
0	x_5	-8/10	0	0	-2/10	(-6/10)	1
c_j -	z_j	112/10	0	0	-7/10	-11/10	0

To drive x_5 from the basis, x_4 should enter (minimum ratio $\phi = 11/6$), as shown in the cut row. Pivoting, the result is still fractional. Using x_2 to generate the next cut,[3] we require that $4/6 - 4/6x_3 - 1/6x_5 \leq 0$.

Tableau 4			-4	-5	0	0	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5	x_6
-5	x_2	$4/6$	0	1	$-2/6$	0	$1/6$	0
-4	x_1	$14/6$	1	0	$2/6$	0	$-4/6$	0
0	x_4	$8/6$	0	0	$2/6$	1	$-10/6$	0
0	x_6	$-4/6$	0	0	$(-4/6)$	0	$-1/6$	1
c_j -	z_j	$76/6$	0	0	$-2/6$	0	$-11/6$	0

[2]Derivation of cut 1:

$$\begin{aligned} x_1 + 2/10x_3 - 4/10x_4 &= 1 + 8/10 \\ x_1 + 2/10x_3 - x_4 + 6/10x_4 &= 1 + 8/10 \\ x_1 - x_4 - 1 &= 8/10 - 2/10x_3 - 6/10x_4 \end{aligned}$$

[3]Derivation of cut 2:

$$\begin{aligned} x_2 - 2/6x_3 + 1/6x_5 &= 4/6 \\ x_2 - x_3 + 4/6x_3 + 1/6x_5 &= 4/6 \\ x_2 - x_3 &= 4/6 - 4/6x_3 - 1/6x_5 \end{aligned}$$

Bringing in x_3 to drive out x_6 ($\phi = \min\{2/4, 11/1\} = 0.5$) reveals the optimal integer solution.

Tableau 5			-4	-5	0	0	0	0
c_B	Basis	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5	x_6
-5	x_2	1	0	1	0	0	$1/4$	$-2/4$
-4	x_1	2	1	0	0	0	$-3/4$	$2/4$
0	x_4	1	0	0	0	1	$-7/4$	$2/4$
0	x_3	1	0	0	1	0	$1/4$	$-6/4$
c_j -	z_j	13	0	0	0	0	$-7/4$	$-2/4$

12. Using the following identities:

$$\begin{aligned} x_3 &= -5 + x_1 + 4x_2 \\ x_4 &= -7 + 3x_1 + 2x_2 \\ x_5 &= -6 + 2x_1 + 2x_2 \end{aligned}$$

each cut can be expressed in terms of x_1 and x_2,

$$\begin{aligned} \text{Cut 1: } 8/10 - 2/10x_3 - 6/10x_4 &\leq 0 \\ \text{becomes: } x_1 + x_2 &\geq 3 \\ \text{Cut 2: } 4/6 - 4/6x_3 - 1/6x_5 &\leq 0 \\ \text{becomes: } x_1 + 3x_2 &\geq 5 \end{aligned}$$

13. See Figure 2.

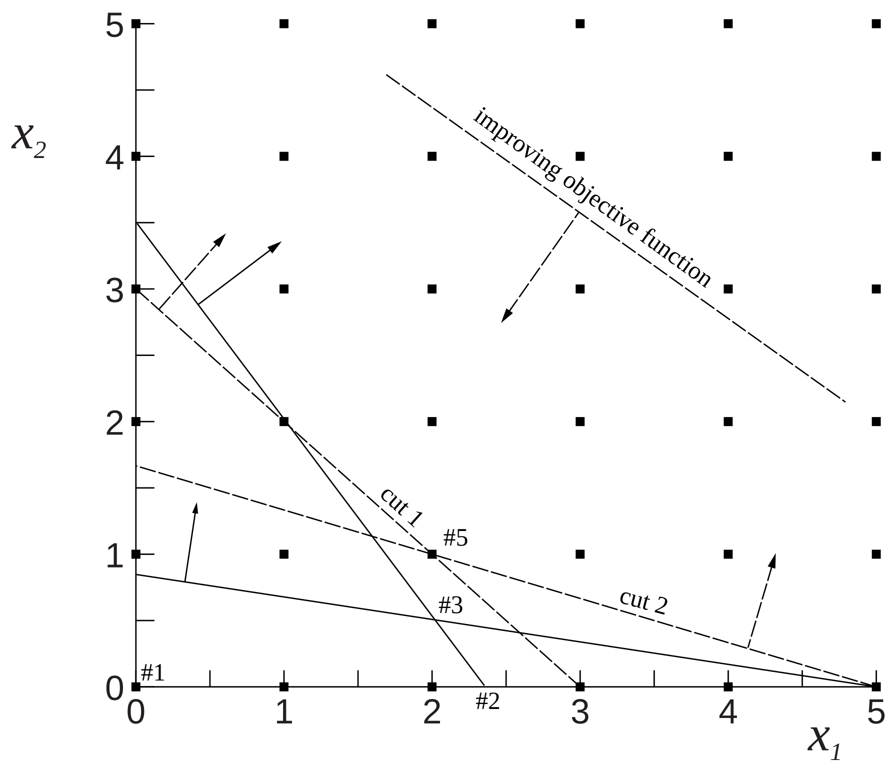

Figure 2: Original feasible region, plus two cuts