Bifurcation theory for a model of the oculomotor neural integrator.

Thomas J. Anastasio* Andrea K. Barreiro† Jared C. Bronski‡

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Abstract

In this paper we consider a linear model for the oculomotor integrator originally proposed by Anastasio and Gad [4]. We are able to use the fact that the system is a finite rank (rank one or two) perturbation of a symmetric, negative definite system, together with a simple idea of classical differential geometry (the envelope of a family of curves) to completely analyze the bifurcations in this model. This analysis gives considerable insight into some observations made in that work. Specifically, we explain how neurons in the cerebellum, which are connected to the neurons that compose the integrator, can sensitively yet independently regulate both the static and dynamic properties of the integrator. Additionally we find that the model has an interesting structural instability. In biologically plausible operating regimes the model lies near a “triple” point, where the regions in which the dominant eigenvalue is real and negative,

*Department of Molecular and Integrative Physiology and Beckman Institute, University of Illinois Urbana-Champaign, Urbana, IL 61820.
†Department of Applied Mathematics, University of Washington, Box 352420, Seattle, WA 98195
‡Department of Mathematics, University of Illinois Urbana-Champaign, 1409 W. Green St., Urbana, IL 61820.
real and positive, and complex all meet. The model provides a new perspective on the possible etiology of a hereditary eye movement disorder known as congenital nystagmus. We believe that the techniques may be applicable to other non-normal systems in biology.

1 Introduction

Non-self-adjoint and non-normal operators often appear in the context of modeling biological systems since the connections in biological networks are frequently not symmetric\(^1\); non-self-adjoint matrices arise in habitat patch networks in metapopulation theory [40], SIR models of infectious diseases [18], and neural networks. Similarly non-self-adjoint differential eigenvalue problems arise in the study of continuum models of population dynamics[21] and activator-inhibitor models of Gierer-Meinhardt type[22, 27]. In the case of neural networks, asymmetry arises because synaptic connections are unidirectional, and the coupling from neuron A to neuron B is typically not the same as the coupling from neuron B to neuron A. Such non-self-adjoint systems admit richer behaviors but the analysis of such systems is typically much more difficult, especially when the system size is large as is typically the case in biology. It is useful to write the non-normal operator as a finite-rank perturbation of a normal (or otherwise well-understood) operator; for example Ovaskainen[40] considers the rank one perturbation that arises when a single habitat is removed from the system and derives approximate expressions for the resulting changes in the eigenvalues. Similarly the study of the stability of spike type solutions to one-dimensional Gierer-Meinhardt models in the limit of slow activator diffusion leads an eigenvalue problem which takes the form of a rank-one perturbation of a standard

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In this paper we focus on a (linear) neural network model of the oculomotor integrator, originally proposed by Anastasio and Gad. The oculomotor integrator is a brainstem structure that integrates velocity signals from oculomotor subsystems (such as the vestibular system, which compensates for head rotations sensed through the semi-circular canals) to produce a position command for the eyes [44]. The integrator is leaky [43] and plastic, adjusting to changes in operational circumstances such as an injury or a new pair of eyeglasses [26, 25, 7]; an hour of exposure to visual/vestibular mismatch (alterations of the normal relationship between head rotation and the apparent rotation of the visual surround) can independently alter the time constant of the integrator and the gain of the overall vestibulo-ocular reflex, each by a factor of two [52]. The integrator is known to maintain connections with cerebellar Purkinje cells [9], which are important for static integrator function [12, 57] and absolutely necessary for plasticity [42, 38]. Experimental results in a variety of vertebrates show that the cerebellum is capable of sensitive and independent control of the gain and the time constant of the integrator. In the language of mathematical modelling, integrator parameters may change in the course of normal operation.

Congenital nystagmus (CN) is a disorder, present at birth or developing early in life, in which the affected individual produces continuous, involuntary movements of the eyes [34, 36]. These eye movements are generally classified as pendular (oscillating) or jerk (exponential fly-aways frequently reset with quick, saccadic eye movements). Different forms can occur in different members of the same family, or even in the same individual at different times [17, 16, 15, 55]. Patients with CN experience reduced visual acuity, but otherwise their oculomotor system seems to be normal [28]. A particular challenge to modelers has been to explain the finding that pendular CN is often observed to be superim-
posed on normal integrator function [16]. Various authors have proposed that
the origin of CN lies in the saccadic [1] or smooth pursuit [28] systems, but
the fact that CN oscillations are continually present makes an explanation that
relies on a malfunction of the integrator, such as [39], particularly apt.

The mathematical model of the oculomotor integrator proposed by Anastasio and Gad [4] is an extension of a model of Cannon et al. [10] to include
cerebellar connections in a way that is consistent with neuroanatomical find-
ings [9]. It is a system of linear differential equations containing two types of
cells, brainstem vestibular neurons and cerebellar Purkinje cells, that integrates
through recurrent feedback (for a review see [44]; for more recent work on recur-
rent feedback models see [47, 48, 31, 24]). The cells are arranged bilaterally [2]
and there are equal numbers of vestibular neurons or Purkinje cells on each side
of the bilateral model. Vestibular neurons are heavily interconnected across the
midline with inhibitory connections. Purkinje cells are relatively few in number,
and there are relatively few vestibular cells which receive input from Purkinje
cells [5]. While the Purkinje cells receive input connections from a large number
of vestibular neurons on both sides of the brainstem, the few output connec-
tions of Purkinje cells are limited to vestibular neurons on the same side only [9].
This leads to a network which is strongly non-symmetric; this asymmetry makes
the behavior of the system surprisingly rich despite its linearity. Plasticity in
this system is created by permitting the strength of the Purkinje-to-vestibular
connections to vary.

We begin by analyzing the dynamics of this model as parameters ($\rho_2$ and $\rho_1$)
representing the Purkinje-to-vestibular connections are varied. The Anastasio-
Gad model has the property that the Purkinje cell to vestibular neuron connec-
tions represent a rank one or rank two perturbation of a system with only real
eigenvalues; this fact allows us to explicitly compute a remarkable amount of
information about the behavior of the spectrum of the matrix. We determine are able to explicitly compute a family of curves which divide the plane into regions with similar qualitative behavior, with the curves themselves representing parameter values for which the model is in some sense structurally unstable. These curves divide the phase plane into regions characterized by the number of complex eigenvalue pairs and the number of eigenvalues in the right half-plane; essentially all qualitative information about the spectrum is computable.

We then apply this understanding of the qualitative behavior of the model to the phenomena of CN. We identify pathological networks in which the integrator, as it attempts to increase its gain, moves into a region where there is an oscillation superimposed on correct integration. We can simulate many other characteristic waveforms of CN as well. We propose a mechanism for CN based on incomplete plasticity; a normal individual can adjust her vestibularto-Purkinje connections freely, whereas a CN patient is constrained to adjust only in certain directions.

We anticipate that these techniques should be applicable to many different types of models which take the form of a small rank perturbation of a self-adjoint problem. For instance, a continuum analog of the model of Anastasio and Gad (see §6) has a similar form to the stability problem for solutions of Gierer-Meinhardt models. Since the bifurcation structure for this continuum model can be completely determined analytically it raises the possibility of improved analysis of models of this type.
2 General Mathematical Considerations: The Role of Non-Normality in Gain

We begin by making some general observations on linear systems, in particular the role played by non-normality in determining gain, which will help to elucidate the behavior of the specific family of models that we consider. There is a rather substantial body of applied mathematics literature on the dynamical properties of non-normal problems - see Trefethen and references therein[53, 54, 19]. One of the important observations is that the spectrum of the problem tells only part of the story: namely the long-time response. In many cases a non-normal operator can exhibit a rather substantial transient growth or gain. In many problems, such as fluid stability, the transient growth due to non-normality has proven to be an impediment to theoretical understanding. In the oculomotor integrator we believe that this phenomenon is exploited to allow correct operation.

A general model for the response of the integrator can be taken to be

\[ \frac{dv}{dt} = Mv + s(t)\vec{b}_{IN} \quad \vec{v}(0) = \vec{0}, \]

where \( \vec{v} \) represents the response of the system (vestibular neurons and Purkinje cells together), \( M \) represents the matrix of connections, \( s(t) \) is the scalar input signal to the integrator, and \( \vec{b}_{IN} \) a fixed vector representing the pattern in which the vestibular neurons receive the input signal. Defining the eigenvectors \( \vec{e}_i \) and adjoint eigenvectors \( \vec{f}_i \) respectively by

\[ M\vec{e}_i = \lambda_i \vec{e}_i \]
\[ M^t \vec{f}_i = \lambda_i \vec{f}_i \]
the response of the system at time \( t \) (assuming \( M \) diagonalizable) is given by
\[
\vec{v}(t) = \sum_i \vec{e}_i \frac{\langle f_i, \vec{b}_{IN} \rangle}{\langle f_i, \vec{e}_i \rangle} \int_0^t e^{\lambda_i (t-t')} s(t') dt'.
\]

For the system to function as an integrator one should have a steady state response in the absence of input: an eigenvalue near zero. Suppose that the following conditions hold

- \( \lambda_1 \) is real, negative, and close to zero.
- The remainder of the eigenvalues \( \lambda_j \) lie in the left half-plane and satisfy
  \[ |\frac{\lambda_j}{\text{Re}(\lambda_j)}| \ll 1 \text{ for } j > 1. \]

Since there is an asymptotic separation of scales \( |\lambda_1| \ll |\text{Re}(\lambda_j)| \) there is an initial transient period after which the response is essentially determined by the dominant eigenvalue \( \lambda_1 \) and corresponding eigenvector \( \vec{e}_1 \) and adjoint eigenvector \( f_1^* \). Specifically if \( \lambda_2 \) is the eigenvalue with next smallest real part then for times \( t \gg 1/|\text{Re}(\lambda_2)| \) we have
\[
\vec{v}(t) \approx \vec{e}_1 \frac{\langle f_1^*, \vec{b}_{IN} \rangle}{\langle f_1, \vec{e}_1 \rangle} \int_0^t e^{\lambda_1 (t-t')} s(t') dt'.
\]

The output of the system should be a linear readout, or superposition, of individual neurons, which we can write as the inner product with some vector \( \vec{b}_{OUT} \).

For the remainder of this paper we assume for simplicity that \( \vec{b}_{IN} = \vec{b}_{OUT} = \vec{b} \): the general case can be treated with only minor modifications. Thus the output of the oculomotor integrator at time \( t \) is given by
\[
\langle \vec{b}, \vec{v}(t) \rangle \approx \| \vec{b} \|^2 \cos(\theta_{be}) \cos(\theta_{bf}) \int_0^t e^{\lambda_1 (t-t')} s(t') dt' \approx g\| \vec{b} \|^2 \int_0^t e^{\lambda_1 (t-t')} s(t') dt'
\]
where \( \cos(\theta_{be}) \) is the cosine of the angle between \( \vec{b} \) and \( \vec{e}_1 \), and similarly for the other two angles. We describe this as an integrator since for times much
shorter than the time-scale determined by the dominant eigenvalue, \( \frac{1}{|\lambda_1|} \gg t \gg \frac{1}{|\text{Re}(\lambda_2)|} \), the above expression is approximately proportional to the integral of the forcing, with constant of proportionality \( g \). We’d like to emphasize the following important fact: In the case where \( M \) is normal (i.e. \( M^4 = MM^4 \)) the problem is always diagonalizable and the eigenvectors and adjoint eigenvectors can be chosen to be the same, \( \vec{e}_i = \vec{f}_i \). In this case the denominator \( \cos(\theta_{ef}) = 1 \). Thus

\[
g = \frac{\cos(\theta_{be}) \cos(\theta_{bf})}{\cos(\theta_{ef})} = \frac{\langle \vec{b}, \vec{e} \rangle \langle \vec{f}, \vec{b} \rangle}{||\vec{b}||^2 \langle \vec{f}, \vec{e} \rangle},
\]

which represents the gain of the system, is at most one. In the non-normal case, however, this denominator can be arbitrarily close to zero, and thus the gain of the system can be made arbitrarily large.

The physical mechanism for this can easily be visualized: suppose that the system has two eigenvectors which point in nearly the same direction, so that the difference is small together with a forcing that points in the direction of the difference between these two eigenvectors. This forcing can be written as a large multiple of one eigenvector minus a large multiple of the other eigenvector. If the eigenvalues are such that one of these eigenvectors decays away more slowly than the other the result is a large response from a small forcing: in effect a gain in the system. This gain can, of course, manifest in any of the modes but since the subdominant modes tend to be heavily damped and since the dynamics of the eyeball-muscle system (the “plant”) acts as a low-pass filter, we are primarily interested in gain only in the dominant mode. We will come back to this in §5, when we analyze the model of the neural integrator.

In the region where \( \cos(\theta_{ef}) \) is small, both large absolute gains are possible, as well as large changes in gain with modest changes in parameter values. This is the optimal region for the integrator to operate, if it is to accomplish the gain ad-
justment documented in experiments [52]. Furthermore, strong non-normality is consistent with neuroanatomy; it is known that the Purkinje cells receive input from many, many vestibular cells, while comparatively few vestibular cells are innervated by Purkinje cells.

This potential for large gain carries with it, however, a potential for instability. It is a simple exercise in linear algebra to check that the inner product between an eigenvector and its adjoint can vanish only under the following two conditions:

• The eigenspace is of dimension at least two.

• The Jordan normal form of $M$ restricted to the eigenspace is not diagonal.

It is well known that a non-diagonal Jordan block is not structurally stable: it represents the boundary between a pair of real eigenvalues and complex conjugate pair of eigenvalues.

Thus we have the following picture: effective operation of the neural integrator represents a compromise between two competing goals. In order to implement substantial changes in gain the system must operate near the bifurcation curve. In effect the system is pushed to the edge of stability. This means that small changes in parameters can also move the integrator into regions of unstable or oscillatory operation which are suggestive of jerk and pendular CN respectively. This will be considered further in §5.

3 The basic model

Following Anastasio and Gad [4] we take a linear model for the oculomotor integrator system. We choose our network to be consistent with the following experimental observations.

O1 The oculomotor integrator is bilaterally symmetric.
O2 The vestibular neuron to Purkinje cell connections are relatively dense.

O3 The Purkinje to vestibular neuron connections are relatively sparse.

O4 There are relatively few Purkinje cells.

For the sake of simplicity we make the following assumptions consistent with the above biological constraints:

A1 The system is bilaterally symmetric, with two Purkinje cells and $N$ vestibular neurons per side.

A2 Only one vestibular neuron is innervated by each Purkinje cell.

A3 Each vestibular neuron has a characteristic decay time and receives input from the nearest neighbors on the contralateral side.

A4 All connections are inhibitory except for the vestibular neuron to Purkinje cell connections, which are excitatory.

A5 We neglect nonlinearity and noise in the system.

The last assumption is obviously only approximately true, and is made for the sake of simplifying the analysis. It would be interesting to study a similar problem with noise and nonlinearity present. A network of the type described above is depicted in Figure (1). Inhibitory connections are denoted by solid lines terminating in a dot(s) at the unit(s) receiving input, while excitatory connections are denoted by solid lines (contralateral input) or dashed lines (ipsilateral input) without terminal dots. Denoting the firing rates of the left and right side vestibular neurons by $v^L_i$ and $v^R_i$ respectively, and similarly the left and right
Figure 1: A typical network. The open circles represent vestibular neurons and the pitchforks Purkinje cells. For simplicity, only a few of the vestibular-to-Purkinje connections are depicted.
side Purkinje cells $P_i^L, P_i^R$ we have the following system governing the dynamics:

\[
\begin{align*}
\frac{dv^L_i}{dt} &= \alpha(-v^L_i - \beta(v^R_{i+1} + v^R_i + v^R_{i-1}) - \rho_1 \delta_{i,k_1} P_1^L - \rho_2 \delta_{i,k_2} P_2^L) \quad (2) \\
\frac{dv^R_i}{dt} &= \alpha(-v^R_i - \beta(v^L_{i+1} + v^L_i + v^L_{i-1}) - \rho_1 \delta_{i,k_1} P_1^R - \rho_2 \delta_{i,k_2} P_2^R) \quad (3) \\
\frac{dP^L_i}{dt} &= \alpha(-P^L_i + \langle \vec{w}^I_i, \vec{v}^L_i \rangle + \langle \vec{w}^C_i, \vec{v}^L_i \rangle) \quad (4) \\
\frac{dP^R_i}{dt} &= \alpha(-P^R_i + \langle \vec{w}^I_i, \vec{v}^R_i \rangle + \langle \vec{w}^C_i, \vec{v}^L_i \rangle). \quad (5)
\end{align*}
\]

Here $\alpha, \beta, \rho_1, \rho_2$ are coupling constants, $\delta_{i,j}$ is the usual Kronecker delta, $k_1$ is the number of the vestibular neuron which is innervated by the first Purkinje cell, and similarly $k_2$. In the figure $k_1 = 1, k_2 = 3$. Finally $\vec{w}^I_i$ and $\vec{w}^C_i$ are vectors representing the connections to the $i^{th}$ Purkinje cell from the ipsilateral (same side) and contralateral (opposite side) neurons respectively.

The above system is invariant under the $\mathbb{Z}_2$ (bilateral) symmetry

\[
(\vec{v}^R, \vec{v}^L, P_1^R, P_1^L, P_2^R, P_2^L) \mapsto (\vec{v}^L, \vec{v}^R, P_1^L, P_1^R, P_2^L, P_2^R)
\]

which switches the left and right sides. For this reason the above system admits a decomposition into the common modes, $\vec{v}^R = \vec{v}^L, P_{1,2}^R = P_{1,2}^L$, where the two side act together, and push-pull modes $\vec{v}^R = -\vec{v}^L, P_{1,2}^R = -P_{1,2}^L$, where the two sides act in opposition. It is the latter that are relevant to integration [43] and in the parameter ranges we have explored the first several dominant eigenvalues are all push-pull modes. In terms of the difference variables $\vec{v} = \vec{v}^R - \vec{v}^L, \vec{P} = P^R - P^L$, the system above becomes

\[
\begin{align*}
\frac{d\vec{v}}{dt} &= \alpha(T\vec{v} - \rho_1 \vec{u}_1 P_1 - \rho_2 \vec{u}_2 P_2) \quad (6) \\
\frac{d\vec{P}_i}{dt} &= \alpha(-P_i + \langle \vec{w}^I_i, \vec{v} \rangle), \quad (7)
\end{align*}
\]

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or, written as a system

$$\frac{d}{dt} \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{w}_1 \\ \vec{w}_2 \end{pmatrix} = \alpha \begin{pmatrix} T & -\rho_1 u_1 & -\rho_2 u_2 \\ \vec{w}_1^t & -1 & 0 \\ \vec{w}_2^t & 0 & -1 \end{pmatrix} \begin{pmatrix} \vec{v} \\ \vec{P}_1 \\ \vec{P}_2 \end{pmatrix} = M(\rho_1, \rho_2) \begin{pmatrix} \vec{v} \\ \vec{P}_1 \\ \vec{P}_2 \end{pmatrix}. \quad (8)$$

The new vestibular neuron to Purkinje cell coupling vectors are given by the difference between the ipsilateral and contralateral vectors, $\vec{w}_i = \vec{w}_i^I - \vec{w}_i^C$, the Purkinje cell to vestibular neuron coupling vectors are given by $(\vec{u}_1)_j = \delta_{j,k_1}, (\vec{u}_2)_j = \delta_{j,k_2}$, and the matrix $T$ representing the effective connections between the vestibular neurons is given by

$$T = \begin{pmatrix} -1 + \beta & \beta & 0 & 0 & \cdots & 0 \\ \beta & -1 + \beta & \beta & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \beta & -1 + \beta \end{pmatrix}. \quad (9)$$

The nearest neighbor interaction for the network of vestibular neurons is convenient, since the eigenvalues are explicitly calculable, though it is not necessary for the analysis. In particular note that the largest eigenvalue of the matrix $T$ above is given by $\lambda_1 = (-1 + \beta(1 + 2 \cos(\frac{\pi}{n+1})))$, where $n$ is the number of vestibular neurons. The parameters $\alpha, \beta$ are set using the following data [57, 12, 5]: the excitation of a solitary vestibular neuron decays with a characteristic time-constant of 5ms, giving $\alpha = 200s^{-1}$. The firing rate for the network of vestibular neurons in the absence of cerebellar input decays with a characteristic time constant of less than one second. Taking a value of 0.2s for the decay of the vestibular network gives $\beta = \frac{39}{40(1+2\cos(\frac{\pi}{5}))}$. We will later consider a six vestibular cell network, giving $\beta = \frac{39}{40(1+2\cos(\frac{\pi}{7}))} \approx .348$. Note that the coupling $\beta$ is of the same order of magnitude as the other parameters in the problem,
which seems biologically plausible.

While we have made a explicit choice of the connectivity matrix $M$ the analysis does not depend in a substantial way on the exact form of the connectivity matrix. It is most convenient if the connections among the vestibular neurons are symmetric and negative definite, and similarly for the connections between the Purkinje cells, and the connections between the vestibular neurons and the Purkinje cells are of rank two, but other than this the analysis requires no real conditions on the model.

4 Analytical Results

As we have been emphasizing this model is non-symmetric: Purkinje cells tend to receive input from many vestibular neurons, while comparatively few vestibular neurons receive input from Purkinje cells. Further, the vestibular neurons excite the Purkinje cells from both sides, while the Purkinje cells inhibit the vestibular neurons on one side only. As a consequence this model can have complex eigenvalues, and can exhibit behavior other than simple growth and decay. We are interested in understanding the spectral properties of the model; in particular

- The number of complex eigenvalues
- The number of eigenvalues in the right half-plane

as $(\rho_2, \rho_1)$ are varied over the plane. Both of these quantities are constant on open sets, with the boundary of these sets being respectively

- The set of $(\rho_2, \rho_1)$ for which $M(\rho_2, \rho_1)$ has a double real eigenvalue
- The set of $(\rho_2, \rho_1)$ for which $M(\rho_2, \rho_1)$ has a purely complex eigenvalue.
Since these sets form the boundary between regions exhibiting qualitatively different behaviors they represent places where the matrix is structurally unstable. While these sets can be calculated by algebraic methods in general the results are not explicit enough to be analytically useful. For instance, the set of \((\rho_2, \rho_1)\) for which \(M(\rho_2, \rho_1)\) has an eigenvalue of multiplicity two or higher (denoted \(\gamma_{bif}\) in this paper) can be computed by first calculating the characteristic polynomial and then computing where the discriminant of the characteristic polynomial vanishes:

\[
\gamma_{bif} = \{ (\rho_2, \rho_1) \mid \text{disc}_\lambda(\det(M(\rho_2, \rho_1) - \lambda I)) = 0 \}.
\]

(See the text of Cox, Little and O’Shea for methods for more information on discriminants[13].) While this can be calculated in principle this curve will be the zero set of some complicated polynomial of high degree in two variables, from which we must then compute the zero set.

Instead we are able to use some of the special structure of this model - the fact that the Purkinje cells represent a low-rank perturbation of a self-adjoint model - to answer these questions completely using some simple ideas from linear algebra and classical differential geometry. This method gives a very explicit parametric representation of these curves, rather than expressing them as the zero set of a polynomial in two variables. Further these ideas work for related continuum models, where it is no longer clear that the discriminant construction makes sense.

Throughout the remainder of the paper we employ the following notation. We let \(M(\rho_2, \rho_1)\) denote the \((n+2) \times (n+2)\) matrix for the full problem and \(T\) denote the \(n \times n\) submatrix corresponding to the vestibular network as defined in (8,9). We also let \(M_0\) denote \(M(0, 0)\) the matrix for the full problem with the weights of the Purkinje cell to vestibular neuron connections set to zero. Finally we will use the wedge notation for Wronskians: for functions \(f_1(\lambda) \ldots f_n(\lambda)\) we
define

\[ f_1 \wedge f_2 \wedge \ldots \wedge f_n = \left| \begin{array}{cccc}
  f_1(\lambda) & f_2(\lambda) & \ldots & f_n(\lambda) \\
  \frac{d f_1}{d \lambda} & \frac{d f_2}{d \lambda} & \ldots & \frac{d f_n}{d \lambda} \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{d^{n-1} f_1}{d \lambda^{n-1}} & \frac{d^{n-1} f_2}{d \lambda^{n-1}} & \ldots & \frac{d^{n-1} f_n}{d \lambda^{n-1}}
\end{array} \right| \]  

(10)

The main goal of this section is to establish that all of the qualitative properties of the system governed by a matrix of the above form can be determined: one can compute explicitly the number of positive and negative eigenvalues, the number of complex eigenvalues in the right and left half-planes, analytically.

### 4.1 Aronszajn-Krein formula

We begin by making a few simple observations. When \((\rho_1 = 0, \rho_2 = 0)\) the problem is block upper-triangular, and thus the eigenvalues are given by the eigenvalues of \(T\) together with an eigenvalue of \(-\alpha\) with multiplicity two, corresponding to the lower right \(2 \times 2\) identity block. The matrix can be written in the form

\[ M = M_0 + \rho_1 M_1 + \rho_2 M_2 \]  

(11)

where the perturbation matrices \(M_{1,2}\) are each of rank one, and thus the full problem is a rank two perturbation of \(M_0\). There are a number of problems in mathematical physics which can be written in the form of a rank-one perturbation of a well-understood problem. Such problems have the useful property that the resolvent of the perturbed problem can be expressed simply in terms of the resolvent of the unperturbed problem, a result known as the Aronszajn-Krein formula. For details on applications of this formula see the lecture notes of Simon [49]. The following lemma is essentially the Aronszajn-Krein formula stated in terms of determinants rather than resolvents.
Lemma 1. The characteristic polynomial of $M$ takes the following form:

$$
\det(M - \lambda I) = (\lambda + \alpha)^2 \det(\alpha T - \lambda I) + P_1(\lambda)\rho_1 + P_2(\lambda)\rho_2 + Q(\lambda)\rho_1\rho_2 \tag{12}
$$

$$
= D(\lambda) + P_1(\lambda)\rho_1 + P_2(\lambda)\rho_2 + Q(\lambda)\rho_1\rho_2 \tag{13}
$$

where $D(\lambda) = (\lambda + \alpha)^2 \det(\alpha T - \lambda I)$ is the determinant of the unperturbed problem, $P_i(\lambda)$ are polynomials of degree (at most) $(N + 1)$ and $Q(\lambda)$ of degree (at most) $N$. In the case where $\vec{w}_1$ and $\vec{w}_2$ (or similarly $\vec{u}_{1,2}$) are linearly dependent $Q(\lambda) = 0$.

Proof. Due to the rank two nature of the perturbation, the characteristic polynomial can contain no powers of $\rho_1$ or $\rho_2$ above the first. The easiest way to see this is via multilinear algebra. The determinant is clearly polynomial in $\lambda, \rho_1, \rho_2$. A general term of the form $\rho_1^j\rho_2^k$ comes from the wedge product of $j$ columns from $M_1$, $k$ columns from $M_2$ and the remainder of the columns $(N + 2 - (j + k))$ from $M_0$. Since $M_{1,2}$ are both rank one any term with $j$ or $k$ greater than one necessarily vanishes. Hence the determinant is of the above form.

Actually calculating the polynomials above requires a little more work. The eigenvalue problem takes the form

$$
\alpha(T\vec{v} - \rho_1P_1\vec{u}_1 - \rho_2P_2\vec{u}_2) = \lambda\vec{v} \tag{14}
$$

$$
(\lambda + \alpha)P_1 = \alpha\langle \vec{w}_1, \vec{v} \rangle \tag{15}
$$

$$
(\lambda + \alpha)P_2 = \alpha\langle \vec{w}_2, \vec{v} \rangle. \tag{16}
$$

Eliminating $P_{1,2}$ gives the following problem for the vestibular cell responses:

$$
(\alpha + \lambda)(\alpha T - \lambda)\vec{v} - \alpha^2\rho_1\langle \vec{w}_1, \vec{v} \rangle\vec{u}_1 - \alpha^2\rho_2\langle \vec{w}_2, \vec{v} \rangle\vec{u}_2 = 0. \tag{17}
$$
Multiplying through by the resolvent of the vestibular network \( R(\lambda) = (\alpha T - \lambda I)^{-1} \) and taking the dot product with \( \vec{w}_1, \vec{w}_2 \) the consistency (eigenvalue) condition

\[
\begin{pmatrix}
(\lambda + \alpha) - \rho_1 \alpha^2 \langle \vec{w}_1, R(\lambda) \vec{u}_1 \rangle & -\rho_2 \alpha^2 \langle \vec{w}_1, R(\lambda) \vec{u}_2 \rangle \\
-\rho_1 \alpha^2 \langle \vec{w}_2, R(\lambda) \vec{u}_1 \rangle & (\lambda + \alpha) - \rho_2 \alpha^2 \langle \vec{w}_2, R(\lambda) \vec{u}_2 \rangle
\end{pmatrix}
\begin{pmatrix}
\langle \vec{w}_1, \vec{v} \rangle \\
\langle \vec{w}_2, \vec{v} \rangle
\end{pmatrix} = 0
\]

(18)

Assuming that \( \langle \vec{w}_1, \vec{v} \rangle \) and \( \langle \vec{w}_2, \vec{v} \rangle \) do not both vanish the above matrix must have zero determinant. If we multiply through by \( \det(\alpha T - \lambda I) \) the eigenvalue condition takes the form

\[
(\lambda + \alpha)^2 \det(\alpha T - \lambda I) + \rho_1 P_1(\lambda) + \rho_2 P_2(\lambda) + Q(\lambda) \rho_1 \rho_2 = 0
\]

(19)

where \( P_i(\lambda), Q(\lambda) \) are given by

\[
P_1(\lambda) = -(\lambda + \alpha)\alpha^2 \langle \vec{w}_1, \text{cof}^t(\alpha T - \lambda I) \vec{u}_1 \rangle
\]

(21)

\[
P_2(\lambda) = -(\lambda + \alpha)\alpha^2 \langle \vec{w}_2, \text{cof}^t(\alpha T - \lambda I) \vec{u}_2 \rangle
\]

(22)

\[
Q(\lambda) = \alpha^4 \det(\alpha T - \lambda I) \begin{vmatrix}
\langle \vec{w}_1, R(\lambda) \vec{u}_1 \rangle & \langle \vec{w}_1, R(\lambda) \vec{u}_2 \rangle \\
\langle \vec{w}_2, R(\lambda) \vec{u}_1 \rangle & \langle \vec{w}_2, R(\lambda) \vec{u}_2 \rangle
\end{vmatrix}
\]

(23)

\[
= \frac{\alpha^4}{\det(\alpha T - \lambda I)} \begin{vmatrix}
\langle \vec{w}_1, \text{cof}^t(\alpha T - \lambda I) \vec{u}_1 \rangle & \langle \vec{w}_1, \text{cof}^t(\alpha T - \lambda I) \vec{u}_2 \rangle \\
\langle \vec{w}_2, \text{cof}^t(\alpha T - \lambda I) \vec{u}_1 \rangle & \langle \vec{w}_2, \text{cof}^t(\alpha T - \lambda I) \vec{u}_2 \rangle
\end{vmatrix}
\]

(24)

(Here \( \text{cof}^t \) denotes the transpose of the cofactor matrix). A short calculation shows that the determinant above has simple poles at the eigenvalues of \( \alpha T \) (assuming the eigenvalues are simple), and thus \( Q(\lambda) \) is a polynomial of the claimed degree. \( \square \)

This construction allows one to compute the eigenvectors, if \( \lambda \) is known.
Given $\lambda$ one can compute the projections $\langle \vec{u}_i, \vec{v} \rangle$ using (18): specifically one can choose

$$\langle \vec{u}_1, \vec{v} \rangle = \alpha^2 \rho_2 \langle \vec{u}_1, \mathbf{R}(\lambda) \vec{u}_2 \rangle$$

(25)

$$\langle \vec{u}_2, \vec{v} \rangle = (\lambda + \alpha) - \alpha^2 \rho_1 \langle \vec{u}_1, \mathbf{R}(\lambda) \vec{u}_1 \rangle$$

(26)

(assuming that these do not both vanish) giving the following expression for the eigenvectors

$$\vec{u} = \frac{\alpha^4 \rho_1 \rho_2 \langle \vec{u}_1, \mathbf{R}(\lambda) \vec{u}_2 \rangle}{\lambda + \alpha} \mathbf{R}(\lambda) \vec{u}_1 + \frac{\alpha^2 \rho_2 (\lambda + \alpha - \alpha^2 \rho_1 \langle \vec{u}_1, \mathbf{R}(\lambda) \vec{u}_1 \rangle)}{\alpha + \lambda} \mathbf{R}(\lambda) \vec{u}_2.$$

Note that these eigenvectors are not normalized. A similar expression holds for the adjoint eigenvectors, with the roles of $\vec{u}_i$ and $\vec{u}_i$ reversed. For a given eigenvalue one can calculate the eigenvectors and eigenvalues and thus the gain defined in §2. For the $\mathbf{T}$ we consider, basically a nearest neighbor Laplacian, the resolvent matrix $\mathbf{R}(\lambda) = (\alpha \mathbf{T} - \lambda \mathbf{I})^{-1}$ can be computed explicitly.

We begin by noting several things about this problem. If we fix a value of $\lambda$ equation (20) gives a condition on the Purkinje-to-vestibular weights $\rho_1, \rho_2$ such that $\lambda$ is an eigenvalue of the model. Thus the Aronszajn-Krein formula gives an explicit one-parameter family of constant eigenvalue curves. For instance, the curve

$$Q(0) \rho_1 \rho_2 + P_z(0) \rho_2 + P_1(0) \rho_1 + D(0) = 0$$

determines the curve in the $\rho_1, \rho_2$ plane (a hyperbola) along which $\lambda = 0$ is an eigenvalue. Similarly for each real $\lambda$ we have a curve in the $\rho_1, \rho_2$ plane along which the given $\lambda$ is an eigenvalue. While we do not, in general, expect to be able to factor the chacteristic polynomial for general $\rho_1, \rho_2$ the fact that we know explicitly the family of constant eigenvalue curves carries a great deal of information about the eigenvalue locations and bifurcations. The information
on eigenvalue coincidence, or bifurcation, is carried by the envelope to the family of curves given in (20).

Definition 1 (Envelope and Bifurcation Curves). In the case where \( Q(\lambda) \) is not identically zero we define the envelope curve \( \gamma_{env} \) to be the union of the pair of curves defined parametrically by

\[
\begin{align*}
\rho_1 &= \frac{-(P_1 \land P_2 + D \land Q) \pm \sqrt{(P_1 \land P_2 - D \land Q)^2 - 4(D \land P_1)(P_2 \land Q)}}{P_1 \land Q} \\
\rho_2 &= \frac{P_1 \land P_2 - D \land Q \mp \sqrt{(P_1 \land P_2 - D \land Q)^2 - 4(D \land P_1)(P_2 \land Q)}}{P_2 \land Q}
\end{align*}
\]

(27) (28)

In the special case \( Q(\lambda) \) is identically zero we define the envelope of the constant eigenvalue lines to be the curve \( \gamma_{env} \) in the \((\rho_2, \rho_1)\) plane defined parametrically by

\[
\begin{align*}
\rho_1 &= -\frac{P_2 \land D(\lambda)}{P_1 \land P_2(\lambda)} \\
\rho_2 &= \frac{P_1 \land D(\lambda)}{P_1 \land P_2(\lambda)}
\end{align*}
\]

(29) (30)

Finally we define the bifurcation curve to be the locus of points in the \((\rho_2, \rho_1)\) plane where \( M(\rho_1, \rho_2) \) has a real eigenvalue of algebraic multiplicity two or higher.

\[
\gamma_{bif} = \{(\rho_2, \rho_1) \mid \exists \lambda \in \mathbb{R} \ s.t. \ \text{dim}(\text{Null}(M(\rho_2, \rho_1) - \lambda I)) \geq 2\}
\]

(31)

The first observation is that, generically, the envelope and the bifurcation curve are the same and thus for each value of \( \lambda \) there is (at most) a pair of points in the \((\rho_2, \rho_1)\) plane for which that \( \lambda \) is a multiple eigenvalue of the model. However in certain non-generic situations there can be a whole curve along which a given eigenvalue has higher multiplicity.
Lemma 2. The envelope curve $\gamma_{env}$ is always contained in the bifurcation curve $\gamma_{bif}$. Generically the bifurcation curve and the envelope curve agree. More specifically: in the case $Q(\lambda)$ is identically zero the bifurcation curve is equal to the envelope curve as long as the condition

$$P_1 \land P_2(\lambda) = P_1 \land D(\lambda) = P_2 \land D(\lambda) = 0$$

do not hold for any real $\lambda$.

In the general case the bifurcation curve is equal to the envelope curve as long as none of the following conditions $C_1, C_2, C_3$ hold for any real $\lambda$:

$$C_1 \quad \text{rank} \begin{pmatrix} D & P_1 & P_2 & Q \\ D' & P_1' & P_2' & Q' \end{pmatrix} < 2 \tag{32}$$

$$C_2 \quad \left\{ \begin{array}{l}
P_1 \land Q(\lambda) \neq 0 \\
P_2 \land Q(\lambda) = 0 \\
P_1 \land P_2(\lambda) = D \land Q(\lambda) \\
D \land P_1(\lambda) = 0 \\
D \land P_2(\lambda) = 0 \\
D \land Q(\lambda) = 0 \end{array} \right. \tag{33}$$

$$C_3 \quad \left\{ \begin{array}{l}
P_2 \land Q(\lambda) \neq 0 \\
P_1 \land Q(\lambda) = 0 \\
P_2 \land P_1(\lambda) = 0 \\
P_2 \land P_2(\lambda) = 0 \\
D \land P_1(\lambda) = D \land Q(\lambda) \end{array} \right. \tag{34}$$

If these genericity conditions fail for some value of $\lambda$ then (generically) there exists a curve in the $(\rho_2, \rho_1)$ plane along which that value of $\lambda$ is a multiple eigenvalue. Note that one can always check whether or not a pair of polynomials have a common root by computing the resultant of the polynomials: one need not be able to explicitly factor the polynomials to test this condition. Thus the genericity condition is readily checkable.
Proof. The basic idea is that the envelope is equal to the bifurcation curve as long as the equations defining the condition for a double eigenvalue are never underdetermined. By “determined” we mean has no more than one solution (in the case $Q(\lambda) = 0$) or no more than two solutions (in the case $Q(\lambda) \neq 0$), so the situation where there is no solution is considered to be determined. In the situation where the equations for a double eigenvalue are underdetermined for a particular value of $\lambda$ one can get a singular piece of the bifurcation curve which is not contained in the envelope.

We first consider the case when $Q(\lambda)$ is identically zero. The condition for an eigenvalue of multiplicity two or higher is

$$D(\lambda) + \rho_1 P_1(\lambda) + \rho_2 P_2(\lambda) = 0 \quad (35)$$

$$D'(\lambda) + \rho_1 P'_1(\lambda) + \rho_2 P'_2(\lambda) = 0. \quad (36)$$

This equation is determined as long as the matrix

$$\begin{pmatrix}
    P_1(\lambda) & P_2(\lambda) & D(\lambda) \\
    P'_1(\lambda) & P'_2(\lambda) & D'(\lambda)
\end{pmatrix}$$

has rank two for all real $\lambda$. This solution gives the envelope curve defined in $(29,30)$. The above matrix is rank-deficient if at least two of the Wronskians $P_1 \wedge P_2, P_1 \wedge D$ and $P_2 \wedge D$ vanish simultaneously. Assuming that the above matrix always has full rank the points on the envelope $(29,30)$ are in one to one correspondence with points at which $(35,36)$ hold, which are exactly the points at which $M$ has a real eigenvalue of higher multiplicity. The envelope is (by construction) tangent to each curve of constant eigenvalue, and this point of tangency represents a point where that particular eigenvalue is degenerate. If this condition fails for some value of $\lambda$ then equations $(35,36)$ are linearly
dependent and thus define a line (assuming that they are consistent). This line represents a “singular” piece of the bifurcation curve for which the equations defining a multiple eigenvalue are satisfied but which do not typically belong to the envelope.

The case where \( Q(\lambda) \) is not identically zero is similar. The envelope satisfies a coupled set of quadratic equations,

\[
D(\lambda) + \rho_1 P_1(\lambda) + \rho_2 P_2(\lambda) + \rho_1 \rho_2 Q(\lambda) = 0
\]

\[
D'(\lambda) + \rho_1 P_1'(\lambda) + \rho_2 P_2'(\lambda) + \rho_1 \rho_2 Q'(\lambda) = 0.
\]

The \( \rho_1 \rho_2 \) terms can be eliminated and one can reduce it to a quadratic in one variable that can, of course, be solved. Analyzing when this procedure produces a unique pair of solutions leads to conditions \( C_1, C_2, C_3 \).

For more details on envelopes see volume III of the text of Spivak [50].

**Lemma 3.** As \((\rho_2, \rho_1)\) are varied so as to cross the envelope the number of real eigenvalues generically changes by two.

**Sketch.** First we consider the case where \( Q(\lambda) \) is identically zero, and the constant eigenvalue curves are lines. It is easy to see that in a neighborhood of any point on the envelope curve with non-vanishing curvature the envelope divides the plane into two regions: one which is multiply (typically doubly) covered by lines of constant eigenvalue, and one which is not covered by any lines of constant eigenvalue. The former is the side which does not contain the center of curvature, while the latter is the side which contains the center of curvature. In the region that is doubly covered by the lines of constant eigenvalue a given point is on two lines, and thus has two distinct real eigenvalues. As this point approaches the envelope the eigenvalues approach each other, and as the point passes through the envelope there are no longer any lines of constant eigen-
value passing through the point, and thus no nearby real eigenvalues. Since
the eigenvalues must vary continuously with the parameters and there are no
nearby real eigenvalues this implies that there must be a nearby complex con-
jugate pair. Thus the envelope of the family of lines of constant eigenvalue also
represents a bifurcation curve where a pair of real eigenvalues bifurcates to a
complex conjugate pair. It is possible that in a degenerate situation there is a
component of the envelope consisting of an isolated point. In this case there
is no bifurcation. This is, however, non-generic. Under perturbation this point
will either disappear or open to a small closed curve (across which there will be
a bifurcation). Similarly it is possible that a portion of the envelope curve is
swept out $k$ times. In this case the number of real eigenvalues will change by $2k$
as the bifurcation curve is crossed. Again this situation will not persist under a
generic perturbation, which will break the $k$-fold covered portion into $k$ distinct
curves.

For non-zero $Q(\lambda)$ the picture is similar, except that the curves of constant
eigenvalue are rational functions and not lines. The general picture is illustrated
in Figure (2), which shows a close-up of the envelope curve from an example
to follow. On one side of the envelope curve there are two constant eigenvalue
curves (in light solid) that intersect each other. This intersection indicates
that there are two nearby real eigenvalues, one corresponding to each constant
eigenvalue curve. As the point of intersection moves toward the envelope curve
(bold solid) the eigenvalues approach each other. At the envelope the eigenvalues
coincide and the constant eigenvalue curve and envelope are tangent. On the
other side of the envelope there are no (nearby) constant eigenvalue curves
through a given point, hence the real eigenvalue pair has bifurcated to a complex
conjugate pair. The main difference from the previous case is that the condition
for determining the direction of the bifurcation is somewhat more complicated,
Figure 2: The bifurcation of eigenvalues across the envelope curve.

and involves the curvatures of the envelope and the constant eigenvalue curves themselves.

We would next like to consider the possibility of eigenvalues of higher multiplicity. The envelope curve is, in general, a well behaved curve and admits a parametrization by arc-length. However this may fail at an isolated set of points in \((\rho_2, \rho_1)\). The next lemma says that (modulo some genericity assumptions) the following are all equivalent, and occur on a codimension two set (isolated points in the \((\rho_2, \rho_1)\) plane):

- Points in the \((\rho_2, \rho_1)\) plane where the model has a real eigenvalue of multiplicity at least three.
- Points where the tangent vector to the envelope curve vanishes.
Cusps in the envelope curve.

**Lemma 4.** The vanishing of the tangent vector to the envelope curve at a point implies that the problem has an eigenvalue of multiplicity (at least) three at that point. The converse holds as long as the following determinant is non-zero at the point in question:

\[
\begin{vmatrix}
P_1 + \rho_2 Q & P_2 + \rho_1 Q \\
P'_1 + \rho_2 Q' & P'_2 + \rho_1 Q'
\end{vmatrix} \neq 0.
\]

Alternatively, the model has a triple eigenvalue if and only if either

\[(P_1 \wedge P_2 \wedge Q)(P_1 \wedge P_2 \wedge D) = (D \wedge P_2 \wedge Q)(P_1 \wedge D \wedge Q) = 0 \\
P_1 \wedge P_2 \wedge Q \neq 0\]

or

\[P_1 \wedge P_2 \wedge Q = P_1 \wedge P_2 \wedge D = D \wedge P_2 \wedge Q = P_1 \wedge D \wedge Q = 0.\]

**Proof.** The conditions for an eigenvalue of multiplicity (at least) two are given by (38,39). Differentiating each of these with respect to \(\lambda\) gives the following equations for \(\rho'_1, \rho'_2:\)

\[\rho'_1(P_1(\lambda) + \rho_2 Q(\lambda)) + \rho'_2(P_2(\lambda) + \rho_1 Q(\lambda)) = 0\]

\[\rho'_1(P'_1(\lambda) + \rho_2 Q'(\lambda)) + \rho'_2(P'_2(\lambda) + \rho_1 Q'(\lambda)) = -(D''(\lambda) + \rho_1 P''_1(\lambda) + \rho_2 P''_2(\lambda) + \rho_1 \rho_2 Q'')\]

The conditions for an eigenvalue of multiplicity at least three are given by (38,39) together with the condition

\[D''(\lambda) + \rho_1 P''_1(\lambda) + \rho_2 P''_2(\lambda) + \rho_1 \rho_2 Q'' = 0. \quad (40)\]
Thus it is clear that $\rho'_1 = 0, \rho'_2 = 0$ implies that the eigenvalue is of multiplicity (at least) three. Further if

$$\begin{vmatrix} P_1 + \rho_2 Q & P_2 + \rho_1 Q \\ P'_1 + \rho_2 Q' & P'_2 + \rho_1 Q' \end{vmatrix} \neq 0$$

then the above system can be solved uniquely for $\rho'_1, \rho'_2$ and the existence of an eigenvalue of multiplicity three implies $\rho'_1 = 0, \rho'_2 = 0$.

A bit more algebra gives another characterization of points where the eigenvalue has multiplicity three or higher. Equations (38,39,40) form a system of three equations in three unknowns $\rho_1, \rho_2$, and $\rho_3 = \rho_1 \rho_2$. Solving these three equations for $(\rho_1, \rho_2, \rho_3)$ and imposing the consistency condition $\rho_3 = \rho_1 \rho_2$ shows that one has a root of multiplicity three if and only if either

$$(P_1 \wedge P_2 \wedge Q) (P_1 \wedge P_2 \wedge D) = (D \wedge P_2 \wedge Q) (P_1 \wedge D \wedge Q)$$

$$P_1 \wedge P_2 \wedge Q \neq 0$$

or all of the Wronskians

$$P_1 \wedge P_2 \wedge Q = P_1 \wedge P_2 \wedge D = D \wedge P_2 \wedge Q = P_1 \wedge D \wedge Q = 0$$

vanish. The first possibility is typically of codimension two - it is expected to occur at isolated values of $\lambda$ corresponding to isolated values of $(\rho_2, \rho_1)$. The second does not typically happen at all, since it requires the simultaneous vanishing of several polynomials. However in Example (2) this case occurs because it is forced by a symmetry of the model.

Finally recall that a simple zero of the tangent vector represents a cusp, and generically an eigenvalue of multiplicity at least three will have multiplicity exactly three, so typically cusps in the envelope are equivalent to triple eigen-
The geometry of a bifurcation in the neighborhood of a triple eigenvalue is illustrated in Figure (8). In a neighborhood of this point there are three dominant eigenvalues which participate in the bifurcation. The cusp of the envelope represents a transition between a bifurcation between the intermediate and the smallest eigenvalue in the trio, and a bifurcation between the intermediate and the largest eigenvalue in the trio. Emerging from the cusp-point is a curve that represents an exchange of dominance phenomenon, with a complex conjugate pair of eigenvalues crossing a single real one.

When considering questions of stability and the behavior of the dominant eigenvalue it is also important to understand the behavior of the complex eigenvalues. In particular one would like to understand the locus of points at which the matrix has purely imaginary eigenvalues, as this curve indicates where the model loses stability due to a complex conjugate pair of eigenvalues crossing from the left half-plane to the right half-plane. In the presence this would lead to the creation of a stable limit cycle via the Hopf bifurcation. For this reason we refer to this locus as the Hopf curve, although we must emphasize that the model we consider here is linear and the classical Andronov-Hopf theorem does not apply.

**Definition 2.** The Hopf curve is the locus of points in the \((\rho_2, \rho_1)\) plane where 

\(M(\rho_2, \rho_1)\) has a pair of purely imaginary eigenvalues. Generically this curve is given parametrically by

\[
\begin{align*}
\text{Re}(D(i\omega)) + \rho_1 \text{Re}(P_1(i\omega)) + \rho_2 \text{Re}(P_2(i\omega)) + \rho_1 \rho_2 \text{Re}(Q(i\omega)) &= 0 \quad (41) \\
\text{Im}(D(i\omega)) + \rho_1 \text{Im}(P_1(i\omega)) + \rho_2 \text{Im}(P_2(i\omega)) + \rho_1 \rho_2 \text{Im}(Q(i\omega)) &= 0, \quad (42)
\end{align*}
\]

where \(\text{Re}, \text{Im}\) represent the real and imaginary parts respectively. The genericity
conditions are the same as in Lemma (2) with the replacement of the Wronskians \( f \wedge g \) by the quantities \( \text{Re}(f(i\omega))\text{Im}(g(i\omega)) - \text{Re}(g(i\omega))\text{Im}(f(i\omega)). \)

The Hopf curve, the envelope, and the \( \lambda = 0 \) eigenvalue curve will all intersect at a single point, the point at which there is a zero eigenvalue of higher multiplicity. With this in mind we complete the analysis of the model presented in the first example.

**Example 1.** We consider the following model:

\[
M = \begin{pmatrix}
-2 & -1 & 0 & -\rho_1 \\
-1 & -2 & -\rho_2 & 0 \\
\sqrt{2} & 1 & -2 & 0 \\
1 & \sqrt{2} & 0 & -2
\end{pmatrix}.
\]

It is straightforward to compute that the characteristic polynomial of this matrix is given by

\[
\det(M - \lambda I) = (1 + \lambda)(2 + \lambda)^2(3 + \lambda) + \left(\lambda^2 + (4 - \sqrt{2})\lambda + (4 - 2\sqrt{2})\right)\rho_1 \\
+ \left(\lambda^2 + (4 - \sqrt{2})\lambda + (4 - 2\sqrt{2})\right)\rho_2 - \rho_1\rho_2
\]

The zero eigenvalue curve is given by

\[
12 + (4 - 2\sqrt{2})\rho_1 + (4 - 2\sqrt{2})\rho_2 - \rho_1\rho_2 = 0
\]

\[
\rho_1 = \frac{12 + (4 - 2\sqrt{2})\rho_2}{\rho_2 - (4 - 2\sqrt{2})}
\]

The bifurcation curve is given by the envelope

\[
\rho_1 = -(\lambda + 2)(\lambda + 2 + \frac{\sqrt{2}}{2}) \pm (\lambda + 2)\sqrt{2(\lambda + \frac{3}{2})(\lambda + \frac{5}{2})}
\]

\[
\rho_2 = -(\lambda + 2)(\lambda + 2 + \frac{\sqrt{2}}{2}) \mp (\lambda + 2)\sqrt{2(\lambda + \frac{3}{2})(\lambda + \frac{5}{2})}
\]
together with the singular piece \( \rho_1 = -\frac{1}{2} \cup \rho_2 = -\frac{1}{2} \), which is associated to the value \( \lambda = -2 + \sqrt{2} \), where the equations defining the the envelope fail to have full rank. The envelope and the singular piece of the bifurcation curve meet tangentially at \((\rho_2 = -\frac{1}{2}, \rho_1 = -\frac{3}{2})\) and \((\rho_2 = -\frac{3}{2}, \rho_1 = -\frac{1}{2})\). Because of the symmetry we have \( P_1 = P_2 \) and thus \( P_1 \wedge P_2 \wedge Q \equiv 0 \), so the condition for a triple eigenvalue reduces to simultaneous vanishing of \( P_1 \wedge D \wedge Q \) and \( D \wedge P_2 \wedge Q \).

Calculating we find that the triple eigenvalue condition becomes

\[
P_1 \wedge D \wedge Q = -16\lambda^3 + (12\sqrt{2} - 96)\lambda^2 + (48\sqrt{2} - 192)\lambda + (46\sqrt{2} - 128) = 0
\]

This cubic has three real roots: a double root at \( \lambda = -2 + \frac{\sqrt{2}}{2} \) and a simple root at \( \lambda = -(\frac{1}{2} + \frac{\sqrt{2}}{4}) \approx 2.35 \). Recall that the envelope is not defined for \( \lambda \in (-\frac{5}{2}, -\frac{3}{2}) \), so the root at \( \lambda = -(\frac{1}{2} + \frac{\sqrt{2}}{4}) \) does not correspond to a real multiple eigenvalue. Thus the only real eigenvalue of multiplicity higher than two is \( \lambda = -2 + \frac{\sqrt{2}}{2} \). Since this eigenvalue is associated to the singular piece of the bifurcation curve we can potentially have many points where this is a triple eigenvalue. Along the curve \( \rho_1 = -\frac{1}{2} \) the eigenvalues are

\[
\lambda = -2 + \frac{\sqrt{2}}{2}, -2 + \frac{\sqrt{2}}{2}, -2 - \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2} - 4\rho_2}{2}.
\]

So the only triple eigenvalue is at \( \rho_2 = -\frac{3}{2} \), the point of intersection with the envelope curve. A similar calculation holds along \( \rho_2 = -\frac{1}{2} \).

The Hopf curve is given parametrically by

\[
\begin{align*}
\rho_1 &= (4 + \sqrt{2})\left(\frac{4\omega^2 - 14}{14}\pm\sqrt{30(18-8\sqrt{2}+(1-2\sqrt{2})\omega^2+\omega^4)}\right) \\
\rho_2 &= (4 + \sqrt{2})\left(\frac{4\omega^2 - 14}{14}\mp\sqrt{30(18-8\sqrt{2}+(1-2\sqrt{2})\omega^2+\omega^4)}\right),
\end{align*}
\]

\( \rho_1 \)
where, as always, the signs are not independent. Note that the argument of the square root is strictly positive, so there exists purely imaginary eigenvalues corresponding to oscillations of any desired frequency. The Hopf curve, the envelope, and the zero eigenvalue line all meet at the points \( (\rho_1 = (4 + \sqrt{2})(-1 \pm \sqrt{30(18 - 8\sqrt{2})}) = 4 + \sqrt{2} \pm \sqrt{30}, \rho_2 = (4 + \sqrt{2})(-1 \mp \sqrt{30(18 - 8\sqrt{2})}) = -(4 + \sqrt{2}) \mp \sqrt{30}. \) The most interesting region of the stability diagram is depicted in Figure (3). The zero eigenvalue curve is depicted in dashed red, the envelope in blue (including a dot at the origin), the singular piece of the bifurcation curve in dot-dashed magenta, and the Hopf curve in solid dotted green.

From this information it is easy to derive the stability diagram. At the origin the eigenvalues are \( \lambda = -3, \lambda = -2, \lambda = -2, \lambda = -1. \) Since this is an isolated point with a degenerate eigenvalue one needs to do a local perturbation analysis near \( \lambda = -2, \rho_1 = 0, \rho_2 = 0 \) to determine if in the neighborhood of this point one has a real pair of eigenvalues or a complex conjugate pair. Letting \( \lambda = -2 + \delta \) shows that near this point one has

\[
\det(M - \lambda I) = -\delta^2 - \sqrt{2}\delta \rho_1 - \delta \sqrt{2} \rho_2 - \rho_1 \rho_2 + O(3),
\]

where \( O(3) \) denotes terms of order three or higher in \( \delta, \rho_i \). The discriminant of the above is \( (\sqrt{2}\rho_1 + \sqrt{2}\rho_2)^2 - 4\rho_1 \rho_2 = 2\rho_1^2 + 2\rho_2^2 > 0, \) indicating that in a neighborhood of the origin the double eigenvalue splits into a real (distinct) pair of eigenvalues. Thus in the region containing the origin and bounded by the singular pieces of the bifurcation curve and the upper branch of the envelope (labelled A) there are four real eigenvalues in the left half-plane. As \( \rho_2 \) is decreased so as to cross the line \( \rho_2 = -\frac{1}{2} \) the first bifurcation occurs. Since this line corresponds to eigenvalue \( \lambda = -2 + \sqrt{2} \) and \( -2 < -2 + \sqrt{2} < -1 \) the bifurcation consists of the two dominant real eigenvalues bifurcating to a complex conjugate pair. Thus in region B we have two real eigenvalues and two complex eigenvalues, all in
the left half-plane. As one leaves region B across the Hopf curve into the region
labelled E the complex conjugate pair moves into the right half-plane, giving two
complex eigenvalues in the right half-plane and two real eigenvalues in the left
half-plane. Proceeding in this fashion the stability diagram can be labelled as
follows:

• Region A: Four real eigenvalues in the left half-plane.
• Region B: Two real and two complex eigenvalues in the left half-plane.
• Region C: Four complex eigenvalues in the left half-plane.
• Region D: Two complex eigenvalues in right half-plane, two complex eigen-
  values in the left half-plane.
• Region E: Two complex eigenvalues in the right half-plane, two real eigen-
  values in the left half-plane.
• Region F: One real eigenvalue in the right half-plane, three real eigenvalues
  in the left half-plane.
• Region G: One real eigenvalue in the right half-plane, one real and two
  complex eigenvalues in the left half-plane.

Additionally there is a narrow region between the regions labelled E and F
(to the left of \( \rho_2 = -(4 + \sqrt{2}) - \sqrt{30} \approx -10.9 \) above the zero eigenvalue curve
and below the envelope curve) where there are two real eigenvalues in the left
half-plane and two real eigenvalues in the right half-plane. This region is not
labelled since it is not visible on this scale.

Note that the one feature which we have not labelled are points where a real
eigenvalue and a complex conjugate pair all have the same real part, correspond-
ing to points where a real eigenvalue and a complex conjugate pair exchange
dominance. It is easy to write down an implicit equation satisfied by these
curves, but it is generally not possible to find an explicit representation as for the Hopf curve, envelope, etc. As we do not need this information to analyze the biological model we do not focus on these curves.

Since the characteristic polynomial of this problem is of order four it would in principle be possible to extract the above information directly from the solution formula for the quartic, though in practice it would be exceedingly difficult to recover such detailed information. In the next section we will consider a model of the oculomotor neural integrator given by a differential equation of order eight. In this situation it is no longer possible even in principle to recover this information from a direct formula for the roots.
5 Application to the oculomotor integrator

As explained in the Introduction, the oculomotor integrator is a neural network that produces eye position commands from the eye velocity commands it receives from oculomotor subsystems such as the vestibular and saccadic systems. Congenital nystagmus (CN) is an eye movement disorder that is characterized by uncontrolled eye movements that are either oscillatory (pendular nystagmus) or unstable, with fly-away movements of exponential profile that are frequently reset by fast eye movements to the center (jerk nystagmus). We will demonstrate that the model for the integrator described in §3 can simulate normal integrator function. Furthermore, instances of this model can show normal integrator function simultaneously with the oscillations characteristic of pendular nystagmus. We are able to make this determination by analysis of the envelope, constant eigenvalue curves and Hopf curve as in §4.

5.1 A network that performs normal integration

An example of an integrator that can perform normal integration with arbitrary adjustment of gain is given by

\[
M = \alpha \begin{pmatrix}
-1 + \beta & \beta & 0 & 0 & 0 & 0 \\
\beta & -1 + \beta & \beta & 0 & 0 & 0 \\
0 & \beta & -1 + \beta & \beta & 0 & 0 \\
0 & 0 & \beta & -1 + \beta & \beta & 0 \\
0 & 0 & 0 & \beta & -1 + \beta & \beta \\
0 & 0 & 0 & 0 & \beta & -1 + \beta \\
-1 & 1 & -1 & 0 & -1 & 0 \\
1 & -1 & 1 & 1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
-\rho_1 & 0 \\
0 & -\rho_2 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
-1 & 0 \\
0 & -1
\end{pmatrix}
\]

(45)
This is the reduced matrix of a model with 6 vestibular neurons and 2 Purkinje cells on each side of the bilaterally symmetric network. The parameter $\alpha$ is set to $200\text{s}^{-1}$ and the parameter $\beta$ is fixed to $\frac{20}{30(1+2\cos(\frac{\pi}{7}))} \approx 0.348$ to give a time constant of $5\text{s}^{-1}$ for the sub-network $T$ in the absence of cerebellar innervation ($\rho_1 = \rho_2 = 0$). The only parameters that may vary are $\rho_1$ and $\rho_2$, the strengths of the Purkinje-to-vestibular connections. In order to fix a certain time constant, $\rho_2$ and $\rho_1$ should be constrained to lie on the appropriate constant eigenvalue curve. The biologically appropriate time constant is in the neighborhood of 20s, so the appropriate eigenvalue is $\lambda = -\frac{1}{20}$. From the results of section 4 we know that this holds along the curve

$$Q\left(-\frac{1}{20}\right)\rho_1\rho_2 + P_2\left(-\frac{1}{20}\right)\rho_2 + P_1\left(-\frac{1}{20}\right)\rho_1 + D\left(-\frac{1}{20}\right) = 0$$

We have computed the polynomials $Q, P_1, P_2, D$ above for this model using
symbolic manipulation. We do not reproduce the results here, since they are rather large (the coefficients are complicated rational functions of $\cos(\pi/7)$) but numerically for $\lambda = -\frac{1}{20}$ the constant eigenvalue curve is given by

$$\rho_1 = \frac{0.137 + 2.536\rho_2}{1 + 0.371\rho_2}$$

This constant eigenvalue curve is tangent to the envelope at the simultaneous solution of

$$0.137 - \rho_1 + 2.536\rho_2 - 0.371\rho_1\rho_2 = 0$$
$$0.577 - \rho_1 + 2.405\rho_2 - 0.473\rho_1\rho_2 = 0.$$  
(Note: we have divided each equation through by a constant.) The biologically important root of the above pair of equations is the one in the first quadrant,
Figure 6: Phase space of network showing normal operation. The colored regions show behavior of the dominant (largest real part) eigenvalue: real unstable (white), complex unstable (light gray), complex stable (gray), real stable (dark gray).

$(\rho_2, \rho_1) = (1.22, 2.23)$ : negative values of $\rho$ would correspond to an excitatory Purkinje to vestibular connection, which is not known to occur.

The basic picture of integrator operation is as follows: let us suppose that $\rho_2$ is allowed to vary, and $\rho_1$ is given by (46), so that $\lambda = -\frac{1}{20}$ is always an eigenvalue. As $\rho_2$ is increased from zero the dominant eigenvalue is fixed at $\lambda = -\frac{1}{20}$ and the (in this case real) subdominant eigenvalue increases. For the reasons outlined in §2 one expects the gain to increase. At $\rho_2 = 1.22$, where the constant eigenvalue curve is tangent to the envelope, there is a collision of eigenvalues and the dominant eigenvalue is degenerate. As $\rho_2$ is further increased the formerly subdominant eigenvalue is now dominant - it is real and larger than $-\frac{1}{20}$. Note that the eigenvalues remain real exactly because the constant eigenvalue curve is tangent to the envelope. If we had passed
through the point \((1.22, 2.23)\) in any other direction this would have crossed
the envelope transversely, and one would have had a pair of real eigenvalues
bifurcating to a complex conjugate pair. Using the results of previous sections we
can compute the gain associated to the dominant mode along the \(\lambda = -\frac{1}{20}\) curve.
In particular using equation (4.1) for the eigenvector together with equation (1)
for the gain we can explicitly compute the gain as a function of \(\rho_2\) along the curve
(46) for which \(\lambda = -\frac{1}{20}\) is an eigenvalue. Again the symbolic representations
of the coefficients are too large to reproduce here but numerically it is given by

\[
g = \frac{\langle f, b \rangle \langle b, e \rangle}{\|b\|^2 \langle f, e \rangle} \approx \frac{0.05(\rho_2 + 1.43)(\rho_2 + 1.86)}{(1.22 - \rho_2)(1.65 + \rho_2)}
\]

Note that, as expected, the denominator of the gain diverges at \(\rho_2 = 1.22\),
where the constant eigenvalue curve is tangent to the envelope and the \(\lambda = -\frac{1}{20}\)
eigenvalue is degenerate.

In Figure (4) we see the response of the network to an impulsive forcing
of the form \(f(t) = \delta(t)(1, 1, 1, 1, 1, 1, 1, 0, 0)^t\) as \(\rho_2, \rho_1\) are varied to increase gain.
(Note that the impulsive forcing is equivalent to free decay with a corresponding
initial condition). Three responses are shown, with \(\rho_2\) set to 0.65, 0.955, and
1.095 respectively. For each value of \(\rho_2, \rho_1\) is set so that the network lies on the
constant time constant curve \(\tau = 20s\). The gains predicted based on a single
dominant mode (1) are 2.52, 5.92 and 12.88. The measured maximum impulse
responses are 30.48, 70.38, 150.35 respectively. The true gains are these numbers
divided by the number of vestibular cells (12), or 2.54, 5.87, 12.53 respectively.
So we see excellent agreement.

Figure (5) displays the interaction of the two dominant eigenvalues of the
network in the vicinity of the current operating region. In order to increase
gain, the network must climb up the \(\lambda = -0.05\) curve in the vicinity of the
double eigenvalue point. At the intersection of this curve with the envelope,
the integrating eigenvalue exchanges dominance with another real eigenvalue, producing an unstable integrator. Note that the larger two gain cases straddle the point where the $\lambda = -\frac{1}{20} s^{-1}$ curve crosses the envelope, indicating an eigenvalue bifurcation in a subdominant mode. In this case it is the mode(s) with the next largest real part. At about $\rho_2 \approx 1.02$ the subdominant complex conjugate pair collides at the real axis, and for $\rho_2$ above this value the first three most dominant eigenvalues are all real. As $\rho_2$ increases the subdominant eigenvalue increases until $\rho_2 \approx 1.22$ where there is an eigenvalue collision and exchange of dominance.

Note that one might expect to see a large response from a mode other than the dominant mode when the constant eigenvalue curve crosses the envelope, around $\rho_2 \approx 1.02$, as this corresponds to a degeneracy in a subdominant eigenvalue. It is important to keep in mind, however, that these modes are very heavily damped and thus much less important than the dominant mode except on very short transient timescales. In this case the eigenvalue collision involves the eigenvalues with the second and third largest real parts. These are a complex conjugate pair for $\rho_2 < 1.02$ and a real pair for $\rho_2 > 1.02$, and the real part at the bifurcation point is about $-23.5$. Since this mode is so heavily damped, compared with the dominant eigenvalue of $-\frac{1}{20}$ the response is still quite small.

Figure (6) is a color-coded diagram showing the character of the dominant eigenvalue. If the network is in error, it may wander into a region where the two eigenvalues are complex, or into the region where one or both eigenvalues are in the right half-plane. In neither case is normal operation possible.

5.2 A network that exhibits pendular nystagmus

A very similar network is given as equation (47); the only change is that the vestibular-to-Purkinje input from a handful of neurons has been altered. How-
ever, as the cerebellum attempts to increase gain by adjusting $\rho_1$ and $\rho_2$, it enters an oscillating regime.

\[
M = \begin{pmatrix}
-1 + \beta & \beta & 0 & 0 & 0 & 0 & -\rho_1 & 0 \\
\beta & -1 + \beta & \beta & 0 & 0 & 0 & 0 & 0 \\
0 & \beta & -1 + \beta & \beta & 0 & 0 & 0 & -\rho_2 \\
0 & 0 & \beta & -1 + \beta & \beta & 0 & 0 & 0 \\
0 & 0 & 0 & \beta & -1 + \beta & \beta & 0 & 0 \\
0 & 0 & 0 & 0 & \beta & -1 + \beta & 0 & 0 \\
-1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \\
1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 \\
\end{pmatrix}
\]

(47)

Figure 7: Impulse response of network as network attempts to increase gain while maintaining $\tau$

We see the impulse response of the network in Figure (7). As in §5.1, $\rho_2$ and $\rho_1$ are varied so as to remain along the $\lambda = -\frac{1}{20}$ curve. As $\rho_2$ (and the gain)
Figure 8: Phase space of network showing pendular nystagmus

increase, the network enters a regime where an oscillation is superimposed on normal integration. Figure (8) illustrates why this behavior occurs. As we follow the $\lambda = -\frac{1}{20} s^{-1}$ curve from left to right, from the edge of the diagram, gain will increase. However, the eigenvalue curve has an intersection with the Hopf curve far to the left of its intersection with the envelope. At this point the integrating eigenvalue exchanges dominance with a pair of complex eigenvalues. Beyond this point, the response of the network contains both the normal integrating mode and a superimposed oscillation. The oscillation decays at a rate that is slow compared to the decay of the integrating mode, and so appears to be steady.

Figure (9) shows the phaspace color-coded by the character of the dominant eigenvalue. Here we can see that the $\lambda = -\frac{1}{20} s^{-1}$ curve travels through a region where a stable complex pair dominates the integrating eigenvalue.
Figure 9: Phase space of network showing pendular nystagmus. The colored regions show behavior of the dominant (largest real part) eigenvalue: real unstable (white), complex unstable (light gray), complex stable (gray), real stable (dark gray).

6 A Related Continuum Model

As a further illustration of the utility of the method we consider a related, and in many ways simpler, continuum model. This can be derived in a straightforward way from the model of Anastasio and Gad by replacing the (relatively numerous) vestibular neurons with a continuum vestibular "line" denoted by $\psi(x)$, while the relatively few Purkinje cells remain discrete. Due to the nearest neighbor form of the bilateral vestibular-vestibular inhibition this goes, in the continuum limit, to a second derivative:

$$\left(\frac{L}{N}\right)^2 \psi_{xx} = \psi_{i+1} + \psi_{i-1} - 2\psi_i + o\left(\left(\frac{L}{N}\right)^3\right)$$
This leads to the following integro-differential eigenvalue problem on $L_2[0, L] \times \mathbb{R}^2$

\[
\left( \frac{L}{N} \right)^2 \psi_{xx} + \alpha \psi + \rho_1 P_1 \delta(x - x_1) + \rho_2 P_2 \delta(x - x_2) = -\lambda \psi, \quad (48)
\]

\[
\psi(0) = 0, \quad \psi(L) = 0
\]

\[
- \alpha P_1 + \int_0^L \psi(x) \phi_1(x) dx = \lambda P_1 \quad (49)
\]

\[
- \alpha P_2 + \int_0^L \psi(x) \phi_2(x) dx = \lambda P_2 \quad (50)
\]

The delta function coupling reflects the sparseness of the Purkinje to vestibular connections, with $x_{1,2}$ denoting the points on the vestibular line innervated by the Purkinje cells, and the functions $\phi_{1,2}(x)$ represent the density of vestibular to Purkinje connections. In the absence of more detailed connection data we take $\phi_{1,2}$ to be constant, although the model can be analyzed in the same manner for essentially arbitrary connection patterns $\phi_{1,2}(x)$. Similarly the Purkinje-vestibular connections $\delta(x - x_i)$ could also be replaced by a more general profile, and a similar analysis would be straightforward.

This model can be solved in much the same way as the discrete model outlined in the previous section. To illustrate we will take $L = 1$, $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{2}$ and the vestibular to Purkinje connections to be $\phi_1(x) = \phi_2(x) = 1$. Algebraically eliminating $P_{1,2}$ from the above model results in the single equation

\[
\left( \frac{L}{N} \right)^2 \psi_{xx} + \alpha \psi + \frac{\rho_1 \phi_1}{\lambda + \alpha} \delta(x - x_1) + \frac{\rho_2 \phi_2}{\lambda + \alpha} \delta(x - x_2) = -\lambda \psi, \quad (51)
\]

\[
ps_i(0) = 0 \quad \psi(L) \quad (52)
\]

As mentioned in the introduction this eigenvalue problem as presented in (51) is very similar to the stability problem for spike solutions to activator-inhibitor models in the limit of slow activator diffusion [20, 27], although the above prob-
lem is somewhat complicated by the fact that the eigenvalues enter in a non-
linear way. Similar models of reaction-diffusion equations with non-local inter-
actions have arisen in a number of other contexts including population dynamics
[21], runaway ohmic heating [11, 32, 33], microwave heating [8] and a reaction-
diffusion equation with a global conservation of mass constraint [46]. We believe
that the techniques presented here should be applicable to understanding these
problems.

Note that when \( \rho_1 = 0, \rho_2 = 0 \), the vestibular neurons decouple from the
Purkinje cells and the problem becomes upper triangular. In this case the
eigenvalues are given by \( \lambda_n = -(\alpha + \frac{n^2 \pi^2}{N^2}) \) and the corresponding eigenvectors
by

\[
\psi_n(x) = \sin(n\pi x)
\]

\[
P_1 = \frac{N^2(1 - \cos(n\pi))}{n^3\pi^3}
\]

\[
P_2 = \frac{N^2(1 - \cos(n\pi))}{n^3\pi^3}
\]

together with \( \lambda = -\alpha \), an eigenvalue of multiplicity two corresponding to the
Purkinje cells. Note that the even modes \( (n = 2k) \) do not actually excite a
Purkinje cell response \( (P_1 = P_2 = 0) \), and thus the even modes are eigenfunc-
tions of this problem for all values of \( \rho_1, \rho_2 \): these modes do not change under
perturbation by the Purkinje cells. Thus we only need track the odd eigenvalues
\( n = 2k + 1 \).

One can apply the techniques developed earlier in the paper to find the
lines of constant eigenvalue. We will not reproduce the entire calculation, but
merely note a few salient points. The first step is to act on Equation 51 with
the resolvent (inverse) operator \( \left( \frac{L^2}{N^2} \partial_{xx} + \alpha + \lambda \right)^{-1} \). The terms \( \left( \frac{L^2}{N^2} \partial_{xx} + \alpha + \lambda \right)^{-1} \delta(x - x_i) \) are simply the Green’s function for the operator \( \left( \frac{L^2}{N^2} \partial_{xx} + \alpha + \lambda \right)^{-1} \).
acting on $L_2[0, L]$ with Dirichlet boundary conditions. This Green’s function can, of course, be easily calculated - see (for example) the text of Keener[30]. Applying the Aronszjan-Krein formula gives the following representation for the constant eigenvalue curves, which are lines since the densities of vestibular to Purkinje connections $P_1(x), P_2(x)$ are linearly dependent

$$\rho_1 \frac{2 \sin(\frac{\omega}{2}) \sin(\frac{3\omega}{2})}{\omega^2} + \rho_2 \frac{2 \sin(\frac{3\omega}{2})^2}{\omega^2} = \frac{\omega^2}{N^2} \cos(\frac{\omega}{2})$$

with $\omega^2 = \frac{N^2(\alpha + \lambda)}{L^2}$. The envelope is found by finding the simultaneous solution to

$$\rho_1 \frac{2 \sin(\frac{\omega}{2}) \sin(\frac{3\omega}{2})}{\omega^2} + \rho_2 \frac{2 \sin(\frac{3\omega}{2})^2}{\omega^2} = \frac{\omega^2}{N^2} \cos(\frac{\omega}{2})$$

This gives the following representation for the bifurcation curve

$$\rho_1 = \frac{3 \omega^3 \csc^4 \left( \frac{\omega}{12} \right) \sec \left( \frac{\omega}{12} \right) \left( 4 \sin \left( \frac{\omega}{12} \right) - 4 \sin \left( \frac{3\omega}{12} \right) + \omega \cos \left( \frac{\omega}{12} \right) \right)}{16N^2}$$

$$\rho_2 = -\frac{\omega^3 \cos^2 \left( \frac{\omega}{N} \right) \csc^4 \left( \frac{\omega}{12} \right) \left( 24 \sin \left( \frac{\omega}{12} \right) - 12 \sin \left( \frac{3\omega}{12} \right) + \omega \cos \left( \frac{\omega}{12} \right) \right)}{2N^2 \left( 2 \cos \left( \frac{\omega}{6} \right) + 1 \right)}$$

The bifurcation curve (envelope) is depicted below for the choice $N = 50$. As can be seen from the figure the origin $(\rho_2, \rho_1) = (0, 0)$ lies on the envelope curve. As noted above in this case the eigenvalues are a dominant eigenvalue of multiplicity two $\lambda = -\alpha$ and a sequence of simple subdominant eigenvalues $\lambda_n = -(\alpha + \frac{n^2 \pi^2}{N^2})$. As one passes into the positive quadrant a pair of real eigenvalues bifurcates from $\lambda \approx -\alpha$ to give a complex conjugate pair of eigenvalues. Additionally there is a triple point, where there is an eigenvalue of multiplicity three near $\rho_2 \approx -2.5, \rho_1 \approx 3$, where the previously subdominant eigenvalue exchanges dominance with the pair of eigenvalues of multiplicity two. One could
Motivated by a biologically interesting example we have developed a bifurcation theory for linear systems which take the form of a low rank perturbation of a known system: in this case the underlying system is symmetric and negative definite. We are able to completely characterize the qualitative behavior of the system by dividing the parameter space into regions according to the nature of the dominant eigenvalue(s). We are able to generate the curves which separate these regions completely analytically without actually having to factor the characteristic polynomial and find the eigenvalues. This is particularly useful in systems which are large dimensional, as is often the case in biology. This identification of various regions of parameter space has led to an improved understanding of the model. In the biological setting discussed here, our analysis allows us to establish that our model is able to simulate both normal and abnormal behavior of the oculomotor integrator. Specifically this model can reproduce the eye position commands associated with pendular congenital nystagmus.

We can hypothesize two origins for persistent oscillations: either an individual is born with a “bad” connection pattern, so that cerebellar control of the integrator forces the eye into a region with oscillations, or an individual is born lacking the required cerebellar control to direct his or her integrator properly. Both of these hypotheses are consistent with findings that genetic abnormalities of the cerebellum, which cause either abnormal development of connections [51], or reduced plasticity in the cerebellum [56], are associated with human CN or with eye movement waveforms in other animals that are similar to those observed in human patients with CN. While we have shown only pendular waveforms, it is straightforward to see that jerk waveforms could be produced.
by an eigenvalue in the right half-plane with a corrective saccade (quick movement) back to the center. Introducing cut-offs into the neuron firing rates (for example, not letting the firing rates go below zero) could allow waveforms to be simulated which depend on gaze direction.

Other authors have hypothesized that a faulty integrator may be the cause of congenital nystagmus. The most common idea is that an instability in the integrator produces slow phase motion away from the “neutral position”, which are then corrected by fast saccades [39]. These models have been criticized on the basis that they are unable to simulate pendular nystagmus, particularly nystagmus that is superimposed on a normal integrating mode [28, 14]. Because these models do not include the asymmetric cerebellar connections, they are not capable of gain adjustment through a linear integrator [10]. As we see here, a non-normal neural network can lead to much richer behavior, including pendular nystagmus that coexists with normal integrator function. The strength of this model is its ability to show how the well-known asymmetry between the brainstem and the cerebellum can account not only for the flexibility of the normal integrator in adapting to changing circumstances, but also for the pathological waveform types observed in CN.

Our integrator model is one of many in the literature based on recurrent excitation. Briefly, firing rates are maintained beyond their characteristic decay time by positive feedback [45, 29, 44, 47, 48, 31, 24] or alternatively, bilateral inhibition [10]. Like all mechanisms based on positive feedback, the neural integrator should be sensitive to the precise values of its positive feedback connection weights. Indirect evidence for this is provided by experiments showing that the gain, time constant, and even the stability of the neural integrator can be adjusted through behavioral paradigms in which the normal relationship between head rotation and eye rotation is disrupted [52, 35]. Additional evidence for
the parametric sensitivity of the real integrator is provided by the observation that seemingly minor factors, such as attention or eye position, can drastically change the waveform of congenital nystagmus [16, 39, 15, 55]. Neural integrator models based on positive feedback (e.g. [10, 3, 4, 6]; this paper) are consistent with these observed sensitivities.

This family of models has been accurately criticized for requiring extremely fine tuning [47]; however sensitivity to synaptic weight values may be not only be tolerable but also desirable, and can be managed by mechanisms that keep them within bounds. Many studies over several decades attest to the importance of the cerebellum in regulating oculomotor variables, including integrator variables (e.g. [42, 37, 23, 41, 38]). This modeling study is the third in a series ([4, 6]) whose aim is to incorporate cerebellar regulation into models of the oculomotor neural integrator, and the main payoff has been a possible explanation for the variety of waveforms observed in congenital nystagmus.

The continuum model analyzed in the final section is very similar to the type of eigenvalue problems which arise in analyzing the stability of solutions to the type of reaction-diffusion models considered in many areas of biology, as well as other areas of science and engineering [27, 21, 8, 20, 22]. We believe that the techniques presented here should be applicable to the study of these reaction diffusion models. We are currently exploring this possibility.

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Figure 10: Envelope curve for the example system, given the choices $N = 50, L = 1, x_1 = 1/3, x_2 = 1/2$